Section 2.3: Monte Carlo Simulation

Discrete-Event Simulation: A First Course

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With *Empirical Probability*, we perform an experiment many times $n$ and count the number of occurrences $n_a$ of an event $\mathcal{A}$.

- The *relative frequency* of occurrence of event $\mathcal{A}$ is $n_a/n$.
- The *frequency theory of probability* asserts that the relative frequency converges as $n \to \infty$.

$$\Pr(\mathcal{A}) = \lim_{{n \to \infty}} \frac{n_a}{n}$$

*Axiomatic Probability* is a formal, set-theoretic approach.

- Mathematically construct the sample space and calculate the number of events $\mathcal{A}$.
- The two are complementary!
Example 2.3.1

- Roll two dice and observe the up faces

\[(1, 1) \quad (1, 2) \quad (1, 3) \quad (1, 4) \quad (1, 5) \quad (1, 6)\\
(2, 1) \quad (2, 2) \quad (2, 3) \quad (2, 4) \quad (2, 5) \quad (2, 6)\\
(3, 1) \quad (3, 2) \quad (3, 3) \quad (3, 4) \quad (3, 5) \quad (3, 6)\\
(4, 1) \quad (4, 2) \quad (4, 3) \quad (4, 4) \quad (4, 5) \quad (4, 6)\\
(5, 1) \quad (5, 2) \quad (5, 3) \quad (5, 4) \quad (5, 5) \quad (5, 6)\\
(6, 1) \quad (6, 2) \quad (6, 3) \quad (6, 4) \quad (6, 5) \quad (6, 6)\]

- If the two up faces are summed, an integer-valued random variable, say \(X\), is defined with possible values 2 through 12

<table>
<thead>
<tr>
<th>Sum, (x)</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Pr(X = x))</td>
<td>(\frac{1}{36})</td>
<td>(\frac{2}{36})</td>
<td>(\frac{3}{36})</td>
<td>(\frac{4}{36})</td>
<td>(\frac{5}{36})</td>
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<td>(\frac{4}{36})</td>
<td>(\frac{3}{36})</td>
<td>(\frac{2}{36})</td>
<td>(\frac{1}{36})</td>
</tr>
</tbody>
</table>

- \(Pr(X = 7)\) could be estimated by replicating the experiment many times and calculating the relative frequency of occurrence of 7's
A *Random Variate* is an algorithmically generated realization of a random variable

- \( u = \text{Random()} \) generates a *Uniform*(0, 1) random variate
- How can we generate a *Uniform*(a, b) variate?

\[
x = a + (b - a)u
\]

```cpp
double Uniform(double a, double b) /* use a < b */ {
    return (a + (b - a) * Random());
}
```
**Equilike Random Variates**

- *Uniform*(0, 1) random variates can also be used to generate an *Equilike*(a, b) random variate

\[
0 < u < 1 \iff 0 < (b - a + 1)u < b - a + 1 \\
\iff 0 \leq \lfloor (b - a + 1)u \rfloor \leq b - a \\
\iff a \leq a + \lfloor (b - a + 1)u \rfloor \leq b \\
\iff a \leq x \leq b
\]

- Specifically, \( x = a + \lfloor (b - a + 1)u \rfloor \)

**Generating an Equilike Random Variate**

```c
long Equilike(long a, long b) /* use a < b */ {
    return (a + (long) ((b - a + 1) * Random()));
}
```
Example 2.3.3 To generate a random variate $x$ that simulates rolling two fair dice and summing the resulting up faces, use

$$x = \text{Equilikely}(1, 6) + \text{Equilikely}(1, 6);$$

Note that this is not equivalent to

$$x = \text{Equilikely}(2, 12);$$

Example 2.3.4 To select an element $x$ at random from the array $a[0], a[1], \ldots, a[n - 1]$ use

$$i = \text{Equilikely}(0, n - 1);$$

$$x = a[i];$$
If three fair dice are rolled, which sum is more likely, a 9 or a 10?

- There are $6^3 = 216$ possible outcomes

\[
Pr(X = 9) = \frac{25}{216} \approx 0.116 \quad \text{and} \quad Pr(X = 10) = \frac{27}{216} = 0.125
\]

- Program `galileo` calculates the probability of each possible sum between 3 and 18

- The drawback of Monte Carlo simulation is that it only produces an estimate
  - Larger $n$ does not guarantee a more accurate estimate
Example 2.3.6

- Frequency probability estimates converge slowly and somewhat erratically

You should *always* run a Monte Carlo simulation with multiple initial seeds
Generate a point at random inside a rectangle with opposite corners at \((\alpha_1, \beta_1)\) and \((\alpha_2, \beta_2)\)

\[x = \text{Uniform}(\alpha_1, \alpha_2);\]
\[y = \text{Uniform}(\beta_1, \beta_2);\]
Generate a point \((x, y)\) at random on the circumference of a circle with radius \(\rho\) and center \((\alpha, \beta)\).

\[
\begin{align*}
\theta &= \text{Uniform}(-\pi, \pi); \\
x &= \alpha + \rho \times \cos(\theta); \\
y &= \beta + \rho \times \sin(\theta);
\end{align*}
\]
Example 2.3.8

- Generate a point \((x, y)\) at random *interior* to the circle of radius \(\rho\) centered at \((\alpha, \beta)\)

\[
\theta = \text{Uniform}(-\pi, \pi);
\]
\[
r = \text{Uniform}(0, \rho);
\]
\[
x = \alpha + r \times \cos(\theta);
\]
\[
y = \beta + r \times \sin(\theta);
\]
Acceptance/Rejection

- Generate a point at random within a circumscribed square and then either accept or reject the point

**Generating a Random Point**

```plaintext
do {
    x = Uniform(-\rho, \rho);
    y = Uniform(-\rho, \rho);
} while (x * x + y * y >= \rho * \rho);

x = \alpha + x;
y = \beta + y;
return (x, y);
```

---

![Diagram showing a random point generation process](image-url)
Buffon’s Needle

- Suppose that an infinite family of infinitely long vertical lines are spaced one unit apart in the \((x, y)\) plane. If a needle of length \(r > 0\) is dropped at random onto the plane, what is the probability that it will land crossing at least one line?

\[
\begin{align*}
\theta & \quad \text{Angle} \\
u & \quad \text{Left-hand endpoint} \\
v & \quad \text{Right-hand endpoint} \\
r & \quad \text{Length of needle}
\end{align*}
\]

- \(u\) is the \(x\)-coordinate of the left-hand endpoint
- \(v\) is the \(x\)-coordinate of the right-hand endpoint, 
  \[v = u + r \cos \theta\]
- The needle crosses at least one line if and only if \(v > 1\)
Program buffon is a Monte Carlo simulation
- The random number library can be used to automatically generate an initial seed

Random Seeding

```
PutSeed(-1); /* any negative integer will do */
GetSeed(&seed); /* trap the value of the initial seed */

. .

printf("with an initial seed of %ld", seed);
```

Inspection of the program buffon illustrates how to solve the problem axiomatically
“Dropped at random” is interpreted (modeled) to mean that $u$ and $\theta$ are independent $\text{Uniform}(0, 1)$ and $\text{Uniform}(-\pi/2, \pi/2)$ r.v.s.
The shaded region has a curved boundary defined by the equation \( u = 1 - r \cos(\theta) \).

If \( 0 < r \leq 1 \), the area of the shaded region is

\[
\pi - \int_{-\pi/2}^{\pi/2} (1 - r \cos \theta) \, d\theta = r \int_{-\pi/2}^{\pi/2} \cos \theta \, d\theta = \cdots = 2r
\]

Therefore, because the area of the rectangle is \( \pi \) the probability that the needle will cross at least one line is \( 2r/\pi \).