CHAPTER 19
PROBLEM 19.1

Determine the maximum velocity and maximum acceleration of a particle which moves in simple harmonic motion with an amplitude of 5 mm and a period of 0.1 s.

SOLUTION

Simple harmonic motion. 
\[ x = x_m \sin(\omega_n t + \phi) \]

where 
\[ \omega_n = \frac{2\pi}{T} = \frac{2\pi}{0.1 \text{ s}} = 62.83 \text{ rad/s} \]
\[ x_m = 5 \text{ mm} = 5 \times 10^{-3} \text{ m} \]

Then 
\[ x = 5 \times 10^{-3} \sin(62.83t + \phi) \]

Differentiate to obtain velocity and acceleration.

\[ v = \dot{x} = \omega_n x_m \cos(\omega_n t + \phi) \]
\[ a = \ddot{x} = -\omega_n^2 x_m \sin(\omega_n t + \phi) \]

**Maximum velocity.**
\[ |\dot{x}_{\text{max}}| = \omega_n x_m = (62.83)(5 \times 10^{-3}) \quad \dot{x}_{\text{max}} = 0.314 \text{ m/s} \]

**Maximum acceleration.**
\[ |\ddot{x}_{\text{max}}| = \omega_n^2 x_m = (62.83)^2(5 \times 10^{-3}) \quad \ddot{x}_{\text{max}} = a_{\text{max}} = 19.74 \text{ m/s}^2 \]
PROBLEM 19.2

Determine the amplitude and maximum velocity of a particle which moves in simple harmonic motion with a maximum acceleration of 60 m/s² and a frequency of 40 Hz.

SOLUTION

Simple harmonic motion. \[ x = x_m \sin(\omega_p t + \phi) \]

where \[ \omega_p = 2\pi f_p = (2\pi)(40) = 80\pi \text{ rad/s} \]

Differentiate to obtain velocity and acceleration.

\[ v = \dot{x} = \omega_p x_m \cos(\omega_p t + \phi) \]

\[ v_m = \omega_p x_m \]

\[ a = \ddot{x} = -\omega_p^2 x_m \sin(\omega_p t + \phi) \]

\[ a_m = \omega_p^2 x_m \]

Use information given to obtain amplitude and maximum velocity.

\[ x_m = \frac{a_m}{\omega_p^2} = \frac{60 \text{ m/s}^2}{(80\pi)^2} = 0.000950 \text{ m} \]

\[ x_m = 0.950 \text{ mm} \]

\[ v_m = \omega_p x_m = (80\pi)(0.000950) = 0.2387 \text{ m/s} \]

\[ v_m = 239 \text{ mm/s} \]
PROBLEM 19.3

A particle moves in simple harmonic motion. Knowing that the amplitude is 300 mm and the maximum acceleration is 5 m/s\(^2\), determine the maximum velocity of the particle and the frequency of its motion.

SOLUTION

Simple harmonic motion.

\[
x = x_m \sin(\omega_n t + \phi) \quad x_m = 0.300 \text{ m}
\]

\[
x = (0.300) \sin(\omega_n t + \phi) \quad (\text{m})
\]

\[
x = (0.3)(\omega_n \cos(\omega_n t + \phi) \quad (\text{m/s})
\]

\[
x = -(0.3)(\omega_n)^2 \sin(\omega_n t + \phi) \quad (\text{m/s})
\]

\[
|a_m| = (0.3 \text{ m/s})(\omega_n)^2 \quad a_m = 5 \text{ m/s}^2
\]

Natural frequency.

\[
\omega_n^2 = \frac{|a_m|}{(0.3 \text{ m})} = \frac{(5 \text{ m/s}^2)}{(0.3 \text{ m})} = 16.667 \text{ rad/s}^2
\]

\[
\omega_n = 4.082 \text{ rad/s}
\]

\[
f_n = \frac{\omega_n}{2\pi}
\]

\[
f_n = \frac{(4.082 \text{ rad/s})}{(2\pi \text{ rad/cycle})} = 0.6497 \text{ Hz}
\]

\[f_n = 0.650 \text{ Hz} \quad \uparrow\]

Maximum velocity.

\[
\nu_m = x_m \omega_n = (0.3 \text{ m})(4.082 \text{ rad/s})
\]

\[\nu_m = 1.225 \text{ m/s} \quad \uparrow\]
PROBLEM 19.4

A 5 kg block is supported by the spring shown. If the block is moved vertically downward from its equilibrium position and released, determine (a) the period and frequency of the resulting motion, (b) the maximum velocity and acceleration of the block if the amplitude of its motion is 50 mm.

SOLUTION

(a) Simple harmonic motion.

\[ x = x_m \sin(\omega_n t + \phi) \]

Natural frequency.

\[ \omega_n = \sqrt{\frac{k}{m}} \quad k = 12000 \text{ N/m} \]

\[ \omega_n = \sqrt{\frac{12000 \text{ N/m}}{15 \text{ kg}}} \]

\[ \omega_n = 20\sqrt{2} = 28.284 \text{ rad/s} \]

\[ \tau_n = \frac{2\pi}{\omega_n} \]

\[ \tau_n = \frac{2\pi}{28.284} = 0.22214 \text{ s} \]

\[ \tau_n = 0.222 \text{ s} \]

\[ f_n = \frac{1}{\tau_n} = \frac{1}{0.22214} = 4.50 \text{ Hz} \]

(b) \[ x_m = 50 \text{ mm} = 0.05 \text{ m} \]

\[ x = 0.05 \sin(28.284t + \phi) \]

Maximum velocity.

\[ v_m = x_m \omega_n = (0.05)(28.284) \]

\[ v_m = 1.414 \text{ m/s} \]

Maximum acceleration.

\[ a_m = x_m \omega_n^2 = (0.05)(28.284)^2 \]

\[ a_m = 40.0 \text{ m/s}^2 \]
**PROBLEM 19.5**

A 32-kg block is attached to a spring and can move without friction in a slot as shown. The block is in its equilibrium position when it is struck by a hammer, which imparts to the block an initial velocity of 250 mm/s. Determine (a) the period and frequency of the resulting motion, (b) the amplitude of the motion and the maximum acceleration of the block.

**SOLUTION**

(a) 

\[ x = x_m \sin(\omega_n t + \phi) \]

\[ \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{12 \times 10^3 \text{ N/m}}{32 \text{ kg}}} = 19.365 \text{ rad/s} \]

\[ \tau_n = \frac{2\pi}{\omega_n} \]

\[ \tau_n = \frac{2\pi}{19.365} \]

\[ \tau_n = 0.324 \text{ s} \]

\[ f_n = \frac{1}{\tau_n} = \frac{1}{0.324} = 3.08 \text{ Hz} \]

(b) At \( t = 0 \), \( x_0 = 0 \),

\[ x_0 = 0 = x_m \sin(\omega_n (0) + \phi) \]

Thus,

\[ \phi = 0 \]

\[ x_0 = v_0 = 250 \text{ mm/s} \]

\[ \dot{x}_0 = v_0 = x_m \omega_n \cos(\omega_n (0) + 0) = x_m \omega_n \]

\[ v_0 = 0.250 \text{ m/s} = x_m (19.365 \text{ rad/s}) \]

\[ x_m = \frac{(0.250 \text{ m/s})}{(19.365 \text{ rad/s})} \]

\[ x_m = 12.91 \times 10^{-3} \text{ m} \]

\[ x_m = 12.91 \text{ mm} \]

\[ a_m = x_m \omega_n^2 = (12.91 \times 10^{-3} \text{ m})(19.365 \text{ rad/s})^2 \]

\[ a_m = 4.84 \text{ m/s}^2 \]
PROBLEM 19.6

A simple pendulum consisting of a bob attached to a cord oscillates in a vertical plane with a period of 1.3 s. Assuming simple harmonic motion and knowing that the maximum velocity of the bob is 400 mm/s, determine (a) the amplitude of the motion in degrees, (b) the maximum tangential acceleration of the bob.

SOLUTION

(a) Simple harmonic motion.

\[ \theta = \theta_m \sin (\omega_n t + \phi) \]

\[ \omega_n = \frac{2\pi}{\tau_n} = \frac{(2\pi)}{1.3 \text{ s}} \]

\[ \omega_n = 4.833 \text{ rad/s} \]

\[ \dot{\theta} = \theta_m \omega_n \cos (\omega_n t + \phi) \]

\[ \dot{\theta}_m = \theta_m \omega_n \]

\[ v_m = l \dot{\theta}_m = l \theta_m \omega_n \]

\[ \theta_m = \frac{v_m}{l \omega_n} \]

For a simple pendulum,

\[ \omega_n = \sqrt{\frac{g}{l}} \]

Thus,

\[ l = \frac{g}{\omega_n^2} = \frac{9.81 \text{ m/s}^2}{(4.833 \text{ rad/s})^2} \]

\[ l = 0.4200 \text{ m} \]

Amplitude.

From Eq. (1),

\[ \theta_m = \frac{v_m}{l \omega_n} = \frac{(0.4 \text{ m/s})}{(0.42 \text{ m})(4.833 \text{ rad/s})} \]

\[ \theta_m = 0.19706 \text{ rad} \]

\[ \theta_m = 11.29^\circ \]

(b) Maximum tangential acceleration.

\[ a_t = l \ddot{\theta} \]

Maximum tangential acceleration occurs when \( \ddot{\theta} \) is maximum.

\[ \ddot{\theta} = -\theta_m \omega_n^2 \sin (\omega_n t + \phi) \]

\[ \ddot{\theta}_{\text{max}} = \theta_m \omega_n^2 \]

\[ (a_t)_{\text{max}} = l \theta_m \omega_n^2 \]

\[ (a_t)_{\text{max}} = (0.4200 \text{ m})(0.19706 \text{ rad})(4.833 \text{ rad/s})^2 \]

\[ (a_t)_m = 1.933 \text{ m/s}^2 \]
PROBLEM 19.7

A simple pendulum consisting of a bob attached to a cord of length \( l = 800 \text{ mm} \) oscillates in a vertical plane. Assuming simple harmonic motion and knowing that the bob is released from rest when \( \theta = 6^\circ \), determine \((a)\) the frequency of oscillation, \((b)\) the maximum velocity of the bob.

\[
\begin{align*}
\omega_n &= \sqrt{\frac{g}{l}} = \sqrt{\frac{(9.81 \text{ m/s}^2)}{(0.8 \text{ m})}} \\
\omega_n &= 3.502 \text{ rad/s} \\
f_n &= \frac{\omega_n}{2\pi} = \frac{(3.502 \text{ rad/s})}{2\pi} \\
f_n &= 0.557 \text{ Hz}
\end{align*}
\]

\[ \theta = \theta_m \sin(\omega_n t + \phi) \]

\[ \theta_m = 6^\circ = 0.10472 \text{ rad} \]

\[
\begin{align*}
\dot{\theta}_m &= \theta_m \omega_n \\
v_m &= l\dot{\theta}_m = l\theta_m \omega_n = (0.8 \text{ m})(0.10472)(3.502) \\
v_m &= 293.4 \times 10^{-3} \text{ m/s} \\
v_m &= 293 \text{ mm/s}
\end{align*}
\]
PROBLEM 19.8

An instrument package $A$ is bolted to a shaker table as shown. The table moves vertically in simple harmonic motion at the same frequency as the variable-speed motor which drives it. The package is to be tested at a peak acceleration of 50 m/s$^2$. Knowing that the amplitude of the shaker table is 60 mm, determine (a) the required speed of the motor in rpm, (b) the maximum velocity of the table.

SOLUTION

In simple harmonic motion,

$$a_{\text{max}} = x_{\text{max}} \omega_n^2$$

$$50 \text{ m/s}^2 = (0.06 \text{ m}) \omega_n^2$$

$$\omega_n^2 = (833.33 \text{ rad/s})^2$$

$$\omega_n = 28.8675 \text{ rad/s}$$

$$f_n = \frac{\omega_n}{2\pi}$$

$$= \frac{28.8675}{2\pi}$$

$$= 4.5945 \text{ Hz (cycles per second)}$$

(a) Motor speed. (4.5945 rev/s)(60 s/min) speed = 276 rpm

(b) Maximum velocity. $v_{\text{max}} = x_{\text{max}} \omega_n = (0.06 \text{ m})(28.8675 \text{ rad/s})$ $v_{\text{max}} = 1.732 \text{ m/s}$
PROBLEM 19.9

The motion of a particle is described by the equation \( x = 5 \sin 2t + 4 \cos 2t \), where \( x \) is expressed in meters and \( t \) in seconds. Determine (a) the period of the motion, (b) its amplitude, (c) its phase angle.

SOLUTION

For simple harmonic motion \( x = x_m \sin(\omega_n t + \phi) \)

Double angle formula (trigonometry): \( \sin(A + B) = (\sin A)(\cos B) + (\sin B)(\cos A) \)

Let \( A = \omega_n t, \quad B = \phi \)

Then \( x = x_m \sin(\omega_n t + \phi) \)
\( x = x_m (\sin \omega_n t)(\cos \phi) + x_m (\sin \phi)(\cos \omega_n t) \)
\( x = (x_m \cos \phi)(\sin \omega_n t) + (x_m \sin \phi)(\cos \omega_n t) \)

Given \( x = 5 \sin 2t + 4 \cos 2t \)
Comparing,
\( \omega_n = 2 \quad x_m \cos \phi = 5 \quad (1) \)
\( x_m \sin \phi = 4 \quad (2) \)

(a) \( \tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{(2 \text{ rad/s})} = \pi \text{ s} \)
\( \tau = 3.14 \text{ s} \)

(b) Squaring Eqs. (1) and (2) and adding,
\( x_m^2 \cos^2 \phi + x_m^2 \sin^2 \phi = 4^2 + 5^2 \)
\( x_m^2 (\cos^2 \phi + \sin^2 \phi) = x_m^2 = 41 \text{ m}^2 \)
\( x_m = 6.40 \text{ m} \)

(c) Dividing Eq. (2) by Eq. (1),
\( \tan \phi = \frac{4}{5} \)
\( \phi = 38.7^\circ \)
PROBLEM 19.10

An instrument package \( B \) is placed on the shaking table \( C \) as shown. The table is made to move horizontally in simple harmonic motion with a frequency of 3 Hz. Knowing that the coefficient of static friction is \( \mu_s = 0.40 \) between the package and the table, determine the largest allowable amplitude of the motion if the package is not to slip on the table. Given answers in both SI and U.S. customary units.

SOLUTION

Maximum allowable acceleration of \( B \).

\[ \sum F = ma : \]

\[ F_m = ma_m \]
\[ \mu_s mg = ma_m \]
\[ a_m = \mu_s g \quad a_m = 0.40 g \]

Simple harmonic motion.

\[ f_n = 3 \text{ Hz} = \frac{\omega_n}{2\pi} \]
\[ \omega_n = 6\pi \text{ rad/s} \]
\[ a_m = x_m \omega_n^2 \]
\[ 0.40g = x_m (6\pi \text{ rad/s})^2 \]
\[ x_m = 1.1258 \times 10^{-3} g \]

Largest allowable amplitude.

SI:
\[ x_m = 1.1258 \times 10^{-3} \times (9.81) = 11.044 \times 10^{-3} \text{ m} \quad x_m = 11.04 \text{ mm} \]
PROBLEM 19.11

A 32-kg block attached to a spring of constant $k = 12 \text{kN/m}$ can move without friction in a slot as shown. The block is given an initial 300-mm displacement downward from its equilibrium position and released. Determine 1.5 s after the block has been released $(a)$ the total distance traveled by the block, $(b)$ the acceleration of the block.

\[ k = 12 \text{kN/m} \]

\[ 32 \text{kg} \]

\[ x = x_m \sin(\omega_n t + \phi) \]

\[ \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{12 \times 10^3 \text{ N/m}}{32 \text{ kg}}} \]

\[ \omega_n = 19.365 \text{ rad/s} \]

\[ \tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{19.365} \]

\[ \tau_n = 0.3245 \text{ s} \]

Initial conditions.

\[ x(0) = 0.3 \text{ m}, \quad \dot{x}(0) = 0 \]

\[ 0.3 = x_m \sin(0 + \phi) \]

\[ \dot{x}(0) = 0 = x_m \omega_n \cos(0 + \phi) \]

\[ \phi = \frac{\pi}{2} \]

\[ x_m = 0.3 \]

\[ x(t) = (0.3)\sin\left(19.365t + \frac{\pi}{2}\right), \quad \tau_n = 0.3245 \text{ s} \]

\[ x(1.5 \text{ s}) = (0.3)\sin\left[(19.365)(1.5) + \frac{\pi}{2}\right] = -0.2147 \text{ m} \]

\[ \dot{x}(1.5 \text{ s}) = (0.3)(19.365)\cos\left[(19.365)(1.5) + \frac{\pi}{2}\right] = 4.057 \text{ m/s} \downarrow \]

In one cycle, block travels $(4)(0.3 \text{ m}) = 1.2 \text{ m}$

To travel 4 cycles, it takes $(4 \text{ cyc})(0.3245 \text{ s/cyc}) = 1.2980 \text{ s}$

at $t = 1.5 \text{ s}$. Thus, total distance traveled is

\[ 4(1.2) + 0.6 + (0.3 - 0.2147) = 5.49 \text{ m} \uparrow \]

\[ \ddot{x}(1.5) = -(0.3)(19.365)^2 \sin\left[(19.365)(1.5) + \frac{\pi}{2}\right] = 80.5 \text{ m/s}^2 \downarrow \]

SOLUTION
PROBLEM 19.12

A 2 kg block is supported as shown by a spring of constant \( k = 400 \text{ N/m} \), which can act in tension or compression. The block is in its equilibrium position when it is struck from below by a hammer, which imparts to the block an upward velocity of 2.5 m/s. Determine \((a)\) the time required for the block to move 100 mm upward, \((b)\) the corresponding velocity and acceleration of the block.

SOLUTION

Simple harmonic motion. 

\[ x = x_m \sin(\omega_n t + \phi) \]

Natural frequency. 

\[ \omega_n = \sqrt{\frac{k}{m}}, \quad k = 400 \text{ N/m} \]

\[ \omega_n = \frac{400 \text{ N/m}}{2} = 10\sqrt{2} = 14.1421 \text{ rad/s} \]

\[ x(0) = 0 = x_m \sin(0 + \phi) \]

\[ \phi = 0 \]

\[ \dot{x}(0) = x_m \omega_n \cos(0 + 0) \]

\[ \dot{x}(0) = 2.5 \text{ m/s} \]

\[ 2.5 = x_m (14.1421) \quad x_m = 0.17678 \text{ m} \]

\[ x = 0.17678 \sin(14.1421t) \text{(m/s)} \] (1)

\((a)\) Time at \( x = 100 \text{ mm} \) \((x = 0.1 \text{ m})\)

\[ t = \frac{\sin^{-1} \left( \frac{0.1}{0.17678} \right)}{14.1421} = 0.04252 \text{ s} \]

\[ t = 0.046 \text{ s} \]

Note: Since \( \omega \) is in rad/s, convert the argument of \( \sin^{-1} \) to radians.

\((b)\) Velocity and acceleration.

\[ \ddot{x} = x_m \omega_n^2 \sin \omega_n t \]

\[ \ddot{x} = 0.04252 \]

\[ \ddot{x} = (0.17678)(14.1421) \cos[(14.1421)(0.04252)] \]

\[ \ddot{x} = 2.0615 \text{ m/s} \]

\[ v = 2.06 \text{ m/s} \uparrow \quad \downarrow \]

\[ \ddot{x} = -(0.1768)(14.1421)^2 \sin[(14.1421)(0.04252)] \]

\[ = -20 \text{ m/s}^2 \]

\[ a = 20.0 \text{ m/s}^2 \downarrow \quad \uparrow \]
PROBLEM 19.13

In Problem 19.12, determine the position, velocity, and acceleration of the block 0.90 s after it has been struck by the hammer.

SOLUTION

Simple harmonic motion. \( x = x_m \sin(\omega_n t + \phi) \)

Natural frequency. \( \omega_n = \sqrt{\frac{k}{m}} \quad k = 400 \text{ N/m} \)
\( \omega_n = \sqrt{\frac{400 \text{ N/m}}{2 \text{ kg}}} \)
\( \omega_n = 10\sqrt{2} = 14.1421 \text{ rad/s} \)

\( x(0) = 0 = x_m \sin(0 + \phi) \)
\( \phi = 0 \)
\( \dot{x}(0) = x_m \omega_n \cos(0 + 0) \quad \dot{x}(0) = 2.5 \text{ m/s} \)
\( 2.5 = x_m (14.1421) \quad x_m = 0.17678 \text{ m} \)
\( x = (0.17678) \sin(14.1421 t) \text{(m/s)} \)

Simple harmonic motion. \( \ddot{x} = x_m \omega_n^2 \sin(\omega_n t + \phi) \)
\( \dot{x} = x_m \omega_n \cos(\omega_n t + \phi) \)
\( \ddot{x} = -x_m \omega_n^2 \sin(\omega_n t + \phi) \)

Note: arguments of \( \sin \) and \( \cos \) are in radians.

At 0.90 s: \( x = (0.17678) \sin[(14.1421)(0.90)] = 0.02843 \text{ m} \quad x = 0.0284 \text{ m} \uparrow \)
\( \dot{x} = (0.17678)(14.1421) \cos[(14.1421)(0.90)] = -2.4675 \text{ m/s} \quad \dot{x} = 2.47 \text{ m/s} \uparrow \)
\( \ddot{x} = -(0.17678)(14.1421)^2 \sin[(14.1421)(0.90)] = -5.6859 \text{ m/s}^2 \quad a = 5.69 \text{ m/s}^2 \downarrow \)
PROBLEM 19.14

The bob of a simple pendulum of length \( l = 800 \text{ mm} \) is released from rest when \( \theta = +5^\circ \). Assuming simple harmonic motion, determine 1.6 s after release (a) the angle \( \theta \), (b) the magnitudes of the velocity and acceleration of the bob.

SOLUTION

\[
\theta = \theta_m \sin(\omega_n t + \phi) \quad \omega_n = \sqrt{\frac{g}{l}} = \sqrt{\frac{9.81 \text{ m/s}}{0.8 \text{ m}}} \quad \omega_n = 3.502 \text{ rad/s}
\]

Initial conditions:

\[
\theta(0) = 5^\circ = \frac{(5)(\pi)}{180} \text{ rad} \\
\dot{\theta}(0) = 0 \\
\theta(0) = \frac{5\pi}{180} = \theta_m \sin(0 + \phi) \\
\dot{\theta}(0) = 0 = \theta_m \omega_n \cos(0 + \phi) \\
\phi = \frac{\pi}{2} \\
\theta_m = \frac{5\pi}{180} \text{ rad} \\
\theta = \frac{5\pi}{180} \sin \left(3.502t + \frac{\pi}{2}\right)
\]

(a) At \( t = 1.6 \text{ s} \):

\[
\theta = \frac{5\pi}{180} \sin \left(3.502(1.6) + \frac{\pi}{2}\right) \quad \theta = 0.06786 \text{ rad} = 3.89^\circ
\]

(b)
PROBLEM 19.14 (Continued)

\[ \dot{\theta} = \theta_m \omega_n \cos(\omega_n t + \phi) = \left( \frac{5\pi}{180} \right)(3.502)\cos\left[ (3.502)(1.6) + \frac{\pi}{2} \right] \]
\[ \dot{\theta}(1.6 \text{ s}) = 0.19223 \text{ rad/s} \]

\[ v = l \dot{\theta} = (0.800 \text{ m})(0.19223 \text{ rad/s}) = 0.1538 \text{ m/s} \]

\[ \ddot{\theta} = -\dot{\theta}_n \omega_n^2 \sin(\omega_n t + \phi) \]
\[ = -\left( \frac{5\pi}{180} \right)(3.502)^2 \sin\left[ (3.502)(1.6) + \frac{\pi}{2} \right] \]
\[ \ddot{\theta} = -0.8319 \text{ rad/s}^2 \]
\[ a = \sqrt{(a_t)^2 + (a_n)^2} \]
\[ a_t = l \ddot{\theta} = (0.8 \text{ m})(-0.8319 \text{ rad/s}^2) = -0.6655 \text{ m/s}^2 \]
\[ a_n = l \ddot{\theta}^2 = (0.8 \text{ m})(0.19223 \text{ rad/s})^2 = 0.02956 \text{ m/s}^2 \]
\[ a = \sqrt{(0.6655)^2 + (0.02956)^2} = 0.6662 \text{ m/s}^2 \]
\[ a = 0.666 \text{ m/s}^2 \]
PROBLEM 19.15

A 5-kg collar rests on but is not attached to the spring shown. It is observed that when the collar is pushed down 180 mm or more and released, it loses contact with the spring. Determine (a) the spring constant, (b) the position, velocity, and acceleration of the collar 0.16 s after it has been pushed down 180 mm and released.

SOLUTION

(a) 
\[ x = x_m \sin(\omega_n t + \phi) \]
\[ x_0 = x_m \sin(0 + \phi) = 0.180 \text{ m} \]
\[ \dot{x}_0 = 0 = x_m \cos(0 + \phi) \]
\[ \phi = \frac{\pi}{2} \]
\[ x_m = 0.180 \text{ m} \]
\[ x = 0.180 \sin\left(\omega_n t + \frac{\pi}{2}\right) \]

When the collar just leaves the spring, its acceleration is \( g \downarrow \) and \( v = 0 \).
\[ \dot{x} = (0.180)\omega_n \cos\left(\omega_n t + \frac{\pi}{2}\right) \]
\[ v = 0 \quad \dot{x} = (0.180)\omega_n \cos\left(\omega_n t + \frac{\pi}{2}\right) \]
\[ \left(\omega_n t + \frac{\pi}{2}\right) = \frac{\pi}{2} \]
\[ a = -g = (-0.180)(\omega_n^2) \sin\left(\omega_n t + \frac{\pi}{2}\right) \]
\[ -g = (-0.180)(\omega_n^2) \quad \omega_n = \sqrt{\frac{9.81 \text{ m/s}^2}{0.180 \text{ m}}} \]
\[ \omega_n = 7.382 \text{ rad/s} \]
\[ \omega_n = \sqrt{\frac{k}{m}} \]
\[ k = m\omega_n^2 \]
\[ = (5 \text{ kg})(7.382 \text{ rad/s})^2 \]
\[ = 272.5 \text{ N/m} \]
\[ k = 273 \text{ N/m} \]
PROBLEM 19.15 (Continued)

\( \omega_n = 7.382 \text{ rad/s} \)

\[ x = 0.180 \sin \left( (7.382)t + \frac{\pi}{2} \right) \]

At \( t = 0.16 \text{ s} \):

**Position.**

\[ x = 0.180 \sin \left[ (7.382)(0.16) + \frac{\pi}{2} \right] = 0.06838 \text{ m} \]

\[ x = 68.4 \text{ mm below equilibrium position} \]

**Velocity.**

\[ \dot{x} = (0.180)(7.382) \cos \left( (7.382)(0.16) + \frac{\pi}{2} \right) \]

\[ = -1.229 \text{ m/s} \]

\[ v = 1.229 \text{ m/s} \]

**Acceleration.**

\[ \ddot{x} = -(0.180)(7.382)^2 \sin \left( (7.382)(0.16) + \frac{\pi}{2} \right) \]

\[ = -3.726 \text{ m/s}^2 \]

\[ a = 3.73 \text{ m/s}^2 \]
PROBLEM 19.16

An 8-kg collar \( C \) can slide without friction on a horizontal rod between two identical springs \( A \) and \( B \) to which it is not attached. Each spring has a constant of 600 N/m. The collar is pushed to the left against spring \( A \), compressing that spring 20 mm, and released in the position shown. It then slides along the rod to the right and hits spring \( B \). After compressing that spring 20 mm, the collar slides to the left and hits spring \( A \), which it compresses 20 mm. The cycle is then repeated. Determine \( a \) the period of the motion of the collar, \( b \) the position of the collar 1.5 s after it was pushed against spring \( A \) and released.

(Note: This is a periodic motion, but not a simple harmonic motion.)

SOLUTION

\( (a) \) For either spring,

\[
\tau_n = \frac{2\pi}{\sqrt{\frac{k}{m}}}
\]

\[
\tau_n = \frac{2\pi}{\sqrt{\frac{600 \text{ N/m}}{8 \text{ kg}}}} = 0.7255 \text{ s}
\]

Complete cycle is \( 1234, 4321 \).

Time from 1 to 2 is \( (\tau_n/4) \), which is the same as time from 3 to 4, 4 to 3 and 2 to 1.

Thus, the time during which the springs are compressed is \( 4(\tau_n/4) = \tau_n = 0.7255 \text{ s} \).

Velocity at 2 or 3.

Use conservation of energy.

\[
v_1 = 0 \quad T_1 = 0 \quad V_1 = \frac{1}{2}kv_1^2 = \frac{1}{2}(600 \text{ N/m})(0.020 \text{ m})^2
\]

\[
V_1 = 0.120 \text{ J}
\]

\[
T_2 = \frac{1}{2}mv_2^2 = \frac{1}{2}(8 \text{ kg})(v_2)^2 \quad T_2 = 4v_2^2
\]

\[
V_2 = 0
\]

\[
T_1 + V_1 = T_2 + V_2 \quad 0 + 0.120 = 4v_2^2 \quad v_2 = 0.1732 \text{ m/s}
\]

Time from 2 to 3 is

\[
t_{2-3} = \frac{(0.020 \text{ m})}{(0.1732 \text{ m/s})} = 0.11545 \text{ s}
\]

and is the same as the time from 3 to 2.

Thus, total time for a complete cycle is

\[
\tau_c = \tau_n + 2t_{2-3}
\]

\[
= 0.7255 + 2(0.11545)
\]

\[
= 0.9564 \quad \tau_c = 0.956 \text{ s}
\]
PROBLEM 19.16 (Continued)

(b) From (a), in 0.9564, the spring A is again fully compressed. Spring B is compressed the second time in 1.5 cycles or \((1.5)(0.9564) = 1.4346\) s. At 1.5 s, the collar is still in contact with spring B moving to the left and is at a distance \(\Delta x\) from the maximum deflection of B equal to

\[
\Delta x = 20 - 20 \cos \left[ \frac{2\pi}{0.7255} (1.5 - 1.4346) \right]
\]

\[
\Delta x = 20 - 16.877
\]

\[
= 3.123 \text{ mm}
\]

Thus, collar C is \(60 - 3.123 = 56.877\) mm from its initial position.

56.9 mm from initial position
PROBLEM 19.17

A 35-kg block is supported by the spring arrangement shown. The block is moved vertically downward from its equilibrium position and released. Knowing that the amplitude of the resulting motion is 45 mm, determine (a) the period and frequency of the motion, (b) the maximum velocity and maximum acceleration of the block.

SOLUTION

(a) Determine the constant $k$ of a single spring equivalent to the three springs

$$ P = k $$

$$ \delta = 16\delta + 8\delta + 8\delta $$

$$ k = 32 \text{ kN/m} $$

Natural frequency.

$$ \omega_n = \sqrt{\frac{k}{m}} $$

$$ = \sqrt{\frac{32 \times 10^3 \text{ N/m}}{35 \text{ kg}}} \quad (1 \text{ N} = 1 \text{ kg} \cdot 1 \text{ m/s}^2) $$

$$ \omega_n = 30.237 \text{ rad/s} $$

$$ \tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{30.23} = 0.208 \text{ s} $$

$$ f_n = \frac{1}{\tau_n} = 4.81 \text{ Hz} $$

(b)

$$ x = x_m \sin(\omega_n t + \phi) \quad x_0 = 0.045 \text{ m} = x_m $$

$$ \omega_n = 30.24 \text{ rad/s} $$

$$ x = 0.045 \sin(30.24t + \phi) $$

$$ \dot{x} = (0.045)(30.24)\cos(30.24t + \phi) $$

$$ \ddot{x} = -(0.045)(30.237)^2 \sin(30.24t + \phi) $$

$$ v_{\text{max}} = 1.361 \text{ m/s} $$

$$ a_{\text{max}} = 41.1 \text{ m/s}^2 $$
PROBLEM 19.18

A 35-kg block is supported by the spring arrangement shown. The block is moved vertically downward from its equilibrium position and released. Knowing that the amplitude of the resulting motion is 45 mm, determine (a) the period and frequency of the motion, (b) the maximum velocity and maximum acceleration of the block.

SOLUTION

(a) Determine the constant \(k\) of a single spring equivalent to the two springs shown.

\[
\delta = \delta_1 + \delta_2 = \frac{R}{16 \text{ kN/m}} + \frac{R}{16 \text{ kN/m}} = \frac{R}{k}
\]

\[
\frac{1}{k} = \frac{1}{16} + \frac{1}{16}
\]

\(k = 8 \text{ kN/m}\)

Period of the motion.

\[
\tau_n = \frac{2\pi}{\sqrt{\frac{m}{L}}} = \frac{2\pi}{\sqrt{\frac{8 \times 10^3}{35}}} = 0.416 \text{ s}
\]

\[
f_n = \frac{1}{\tau_n} = \frac{1}{0.416} = 2.41 \text{ Hz}
\]

(b) \(\omega_n = 2\pi f_n = 2\pi(2.41) = 15.12 \text{ rad/s}\)

\(x = 0.045 \sin(15.12t + \phi)\)

\[
\dot{x} = (0.045)(15.12) \cos(15.12t + \phi)
\]

\[
\ddot{x} = -(0.045)(15.12)^2 \sin(15.12t + \phi)
\]

\(v_{\text{max}} = 0.680 \text{ m/s}\)

\(a_{\text{max}} = 10.29 \text{ m/s}^2\)
PROBLEM 19.19

A 15 kg block is supported by the spring arrangement shown. If the block is moved from its equilibrium position 40 mm vertically downward and released, determine (a) the period and frequency of the resulting motion, (b) the maximum velocity and acceleration of the block.

SOLUTION

Determine the constant $k$ of a single spring equivalent to the three springs shown.

Springs 1 and 2:

\[ \delta = \delta_1 + \delta_2, \quad \text{and} \quad \frac{P_1}{k'} = \frac{P_1}{k_1} + \frac{P_1}{k_2} \]

Hence,

\[ k' = \frac{k_1 k_2}{k_1 + k_2} \]

where $k'$ is the spring constant of a single spring equivalent of springs 1 and 2.

Springs $k'$ and 3 Deflection in each spring is the same.

So

\[ P = P_1 + P_2, \quad \text{and} \quad P = k \delta, \quad P_1 = k' \delta, \quad P_2 = k_3 \delta \]

Now

\[ k \delta = k' \delta + k_3 \delta \]

\[ k = k' + k_3 = \frac{k_1 k_2}{k_1 + k_2} + k_3 \]

\[ k = \frac{(4)(2.4)}{4 + 2.4} + 3.2 = 4.7 \text{ kN/m} = 4700 \text{ N/m} \]

\[ m = 15 \text{ kg} \]

\[ \omega_n^2 = \frac{k}{m} = \frac{4700}{15} = 313.33 \quad \omega_n = 17.7012 \text{ rad/s} \]
PROBLEM 19.19 (Continued)

(a) Period.
\[ \tau_n = \frac{2\pi}{\omega_n} \]
\[ \tau_n = 0.355 \text{ s} \]

Frequency.
\[ f_n = \frac{\omega_n}{2\pi} \]
\[ f_n = 2.82 \text{ Hz} \]

(b) Simple harmonic motion.
\[ x = 0.04 \sin(\omega t + \phi) \text{ in m} \]
\[ x_m = 40 \text{ mm} \]

\[ \dot{x} = (0.04)(17.7012)\cos(\omega t + \phi) \]

\[ v_m = \dot{x}_m = 0.70805 \text{ m/s} \]
\[ v_m = 0.708 \text{ m/s} \]

\[ \ddot{x} = -(0.04)(17.7012)^2 \sin(\omega t + \phi) \]

\[ a_m = \ddot{x}_m = 12.533 \text{ m/s}^2 \]
\[ a_m = 12.53 \text{ m/s}^2 \]
**PROBLEM 19.20**

A 5-kg block, attached to the lower end of a spring whose upper end is fixed, vibrates with a period of 6.8 s. Knowing that the constant $k$ of a spring is inversely proportional to its length, determine the period of a 3-kg block which is attached to the center of the same spring if the upper and lower ends of the spring are fixed.

**SOLUTION**

Equivalent spring constant.

\[ k' = 2k + 2k = 4k \]

(Deflection of each spring is the same.)

For case (1),

\[ \tau_{n1} = 6.8 \text{ s} \]
\[ \omega_{n1} = \frac{2\pi}{\tau_{n1}} = \frac{2\pi}{6.8} = 0.924 \text{ rad/s} \]
\[ \omega_{n1}^2 = \frac{k}{m_1} \]
\[ k = m_1\omega_{n1}^2 = (5)(0.924)^2 = 4.2689 \text{ N/m} \]

For case (2),

\[ \omega_{n2}^2 = \frac{4k}{m_2} = \frac{(4)(4.2689)}{3} = 5.6918 \text{ (rad/s)}^2 \]
\[ \omega_{n2} = 2.3857 \text{ rad/s} \]
\[ \tau_{n2} = \frac{2\pi}{\omega_{n2}} = \frac{2\pi}{2.3857} \]
\[ \tau_{n2} = 2.63 \text{ s} \]

\[ \tau_{n2} = 2.63 \text{ s} \]
PROBLEM 19.21

A 15 kg block is supported by the spring arrangement shown. The block is moved from its equilibrium position 20 mm vertically downward and released. Knowing that the period of the resulting motion is 1.5 s, determine (a) the constant \( k \), (b) the maximum velocity and maximum acceleration of the block.

SOLUTION

Since the force in each spring is the same, the constant \( k' \) of a single equivalent spring is

\[
\frac{1}{k'} = \frac{1}{2k} + \frac{1}{k} + \frac{1}{k} \quad \text{and} \quad k' = \frac{k}{2.5} \quad \text{(See Problem 19.19.)}
\]

(a)

\[
\tau_n = 1.5 \text{ s} = \frac{2\pi}{\sqrt{m'k'}}; \quad k = \left( \frac{2\pi}{1.5} \right)^2 (15 \text{ kg})(2.5)
\]

\[
k = \frac{(2\pi)^2}{(1.5 \text{ s})^2} (15 \text{ kg})(2.5)
\]

\[
= 657.974 \text{ N/m}
\]

\( k = 658 \text{ N/m} \uparrow \)

(b)

\[
x = x_m \sin(\omega_n t + \phi)
\]

\[
\dot{x} = x_m \omega_n \cos(\omega_n t + \phi)
\]

\[
v_{\text{max}} = x_m \omega_n
\]

\[
\omega_n = \frac{2\pi}{\tau_n} = \frac{2\pi}{1.5} = 4.189 \text{ rad/s}
\]

\[
x_m = 20 \text{ mm} = 0.02 \text{ m}
\]

\[
v_{\text{max}} = (0.02 \text{ m})(4.189 \text{ rad/s}) \quad v_{\text{max}} = 0.0838 \text{ m/s} \uparrow
\]

\[
\ddot{x} = -x_m \omega_n^2 \cos(\omega_n t + \phi)
\]

\[
|a_{\text{max}}| = x_m \omega_n^2 = (0.02 \text{ m})(4.189)^2 \quad |a_{\text{max}}| = 0.351 \text{ m/s}^2 \uparrow
\]
**PROBLEM 19.22**

Two springs of constants $k_1$ and $k_2$ are connected in series to a block $A$ that vibrates in simple harmonic motion with a period of 5 s. When the same two springs are connected in parallel to the same block, the block vibrates with a period of 2 s. Determine the ratio $k_1/k_2$ of the two spring constants.

**SOLUTION**

Equivalent springs.

Series:

$$k_s = \frac{k_1k_2}{k_1 + k_2}$$

Parallel:

$$k_p = k_1 + k_2$$

$$\frac{\tau_s}{\tau_p} = \left(\frac{5}{2}\right)^2 = \frac{k_p}{k_s} = \frac{k_1 + k_2}{\frac{(k_1k_2)}{k_1 + k_2}} = \left(\frac{k_1 + k_2}{k_1k_2}\right)^2$$

$$\frac{(6.25)(k_1k_2)}{k_1} = \frac{k_1^2 + 2k_1k_2 + k_2^2}{k_1}$$

$$k_1 = \frac{(4.25)k_2 + \sqrt{(4.25)^2k_2^2 - 4k_2^2}}{2}$$

$$\frac{k_1}{k_2} = 2.125 \pm \sqrt{3.516} = 0.250 \text{ or } 4.00$$

$$\frac{k_1}{k_2} = 0.250 \text{ or } 4.00$$
PROBLEM 19.23

The period of vibration of the system shown is observed to be 0.6 s. After cylinder $B$ has been removed, the period is observed to be 0.5 s. Determine (a) the weight of cylinder $A$, (b) the constant of the spring.

![Diagram]

**SOLUTION**

\[
\begin{align*}
    m_1 &= m_A + 1.5 \quad \tau_1 = 0.6 \text{ s} \\
    \tau_1 &= \frac{2\pi}{\omega_1} \quad \omega_1 = \frac{2\pi}{\tau_1} = \frac{2\pi}{0.6} = 3.333 \text{ rad/s} \\
    \omega_1^2 &= k \quad m_1\omega_1^2 = m_A + 1.5(3.333)^2 \quad (1) \\
    m_2 &= m_A \quad \tau_2 = 0.5 \text{ s} \\
    \tau_2 &= \frac{2\pi}{\omega_2} \quad \omega_2 = \frac{2\pi}{\tau_2} = \frac{2\pi}{0.5} = 4\pi \text{ rad/s} \\
    \omega_2^2 &= k \quad m_2\omega_2^2 = m_A(4\pi)^2 \quad (2)
\end{align*}
\]

(a) Equating the expressions found for $k$ in Eqs. (1) and (2):

\[
m_A + 1.5(3.333)^2 = m_A(4\pi)^2
\]

\[
(11.111)(m_A + 1.5) = 16m_A
\]

\[
4.889m_A = 16.6665
\]

\[
m_A = 3.40898 \text{ kg}
\]

\[
W_A = m_Ag = (3.40898)(9.81) 
W_A = 33.4 \text{ N}
\]

(b) Eq. (1):

\[
k = 3.40898 + 1.5(3.333) \text{ rad/s}^2
\]

\[
k = 538.22 \text{ N/m} 
\]

\[
k = 538 \text{ N/m}
\]
PROBLEM 19.24

The period of vibration of the system shown is observed to be 0.8 s. If block \( A \) is removed, the period is observed to be 0.7 s. Determine (a) the mass of block \( C \), (b) the period of vibration when both blocks \( A \) and \( B \) have been removed.

SOLUTION

\[ m_1 = m_C + 6 \text{ kg} \quad \tau_1 = 0.8 \text{ s} \]

\[ W_1 = \frac{2\pi}{\tau_1} = \frac{2\pi}{0.8} \text{ rad/s} \]

\[ \omega_1^2 = \frac{k}{m_1} \quad k = m_1\omega_1^2 = (m_C + 6)\left(\frac{2\pi}{0.8}\right)^2 \quad (1) \]

\[ m_2 = m_C + 3 \text{ kg} \quad \tau_2 = 0.7 \text{ s} \]

\[ \omega_2^2 = \frac{k}{m_2} \quad k = m_2\omega_2^2 = (m_C + 3)\left(\frac{2\pi}{0.7}\right)^2 \quad (2) \]

Equating the expressions found for \( k \) in Eqs. (1) and (2):

\[ (m_C + 6)\left(\frac{2\pi}{0.8}\right)^2 = (m_C + 3)\left(\frac{2\pi}{0.7}\right)^2 \]

\[ \frac{m_C + 6}{m_C + 3} = \left(\frac{0.8}{0.7}\right)^2 ; \text{ solve for } m_C : \quad m_C = 6.80 \text{ kg} \]

\[ \omega_3 = \frac{2\pi}{\tau_3} \]

\[ \omega_3^2 = \frac{k}{m_C} \quad k = m_C\omega_3^2 = m_C\left(\frac{2\pi}{\tau_3}\right)^2 \quad (3) \]

Equating expressions for \( k \) from Eqs. (2) and (3),

\[ (m_C + 3)\left(\frac{2\pi}{0.7}\right)^2 = m_C\left(\frac{2\pi}{\tau_3}\right)^2 \]

Recall \( m_C = 6.8 \text{ kg} : \)

\[ (6.8 + 3)\left(\frac{2\pi}{0.7}\right)^2 = 6.8 \left(\frac{2\pi}{\tau_3}\right)^2 \]

\[ \left(\frac{\tau_3}{0.7}\right)^2 = \frac{6.8}{9.8} = 0.833 \quad \tau_3 = 0.583 \text{ s} \]
**PROBLEM 19.25**

The period of vibration of the system shown is observed to be 0.2 s. After the spring of constant \( k_2 = 4 \text{ kN/m} \) is removed and block \( A \) is connected to the spring of constant \( k_1 \), the period is observed to be 0.12 s. Determine (a) the constant \( k_1 \) of the remaining spring, (b) the weight of block \( A \).

**SOLUTION**

Equivalent spring constant for springs in series.

\[
k_e = \frac{k_1 k_2}{(k_1 + k_2)}
\]

For \( k_1 \) and \( k_2 \),

\[
\tau = \frac{2\pi}{\sqrt{\frac{k_e}{m}}}
\]

For \( k_1 \) alone,

\[
\tau' = \frac{2\pi}{\sqrt{\frac{k_1}{m}}}
\]

(a)

\[
k_2 \left( \frac{\tau}{\tau'} \right)^2 = k_1 + k_2
\]

\[
\frac{\tau}{\tau'} = \frac{0.2}{0.12} = 1.6667
\]

\[
k_2 = 4 \text{ kN/m}
\]

\[
(4 \text{ kN/m})(1.6667 \text{ s})^2 = k_1 + 4 \text{ kN/m}
\]

\[
k_1 = 7.11 \text{ kN/m}
\]

(b)

\[
k_1 = 7.1111 \text{ kN/m}
\]

\[
\tau' = \frac{2\pi}{\sqrt{\frac{k_1}{m}}}
\]

\[
m_A = \frac{(\tau')^2 k_1}{(2\pi)^2}; \quad W_A = m_A g
\]

\[
k_1 = 7.1111 \text{ kN/m} = 7111.1 \text{ N/m}
\]

\[
W_A = \frac{(9.81 \text{ m/s}^2)(7111.1 \text{ N/m})}{(2\pi)^2} \quad W_A = 25.4 \text{ N}
\]
PROBLEM 19.26

The 50-kg platform $A$ is attached to springs $B$ and $D$, each of which has a constant $k = 2$ kN/m. Knowing that the frequency of vibration of the platform is to remain unchanged when a 40-kg block is placed on it and a third spring $C$ is added between springs $B$ and $D$, determine the required constant of spring $C$.

SOLUTION

Frequency of the original system.

Springs $B$ and $D$ are in parallel.

$$k_e = k_B + k_D = 2(2 \text{ kN/m}) = 4 \text{ kN/m} = 4000 \text{ N/m}$$

$$\omega_n^2 = \frac{k_e}{m_A} = \frac{4000 \text{ N/m}}{50 \text{ kg}}$$

$$\omega_n^2 = 80 \text{(rad/s)}^2$$

Frequency of new system.

Springs $A$, $B$, and $C$ are in parallel.

$$k'_e = k_B + k_D + k_C = (2)(200) + k_C \text{ in N/m}$$

$$(\omega'_n)^2 = \frac{k'_e}{m_A + m_B} = \frac{(2000 + k_c)}{(50 \text{ kg} + 40 \text{ kg})}$$

$$(\omega'_n)^2 = \frac{(2000 + k_c)}{90}$$

$$\omega_n^2 = (\omega'_n)^2$$

$$80 = \frac{(2000 + k_c)}{90}$$

$$k_C = 5200 \text{ N/m} = 5.2 \text{ kN/m} \quad k_C = 5.20 \text{ kN/m}$$
PROBLEM 19.27

From mechanics of materials it is known that when a static load \( P \) is applied at the end \( B \) of a uniform metal rod fixed at end \( A \), the length of the rod will increase by an amount \( \delta = \frac{PL}{AE} \), where \( L \) is the length of the undeformed rod, \( A \) is its cross-sectional area, and \( E \) is the modulus of elasticity of the metal. Knowing that \( L = 450 \) mm and \( E = 200 \) GPa and that the diameter of the rod is 8 mm, and neglecting the mass of the rod, determine \((a)\) the equivalent spring constant of the rod, \((b)\) the frequency of the vertical vibrations of a block of mass \( m = 8 \) kg attached to end \( B \) of the same rod.

SOLUTION

\((a)\)

\[ P = k_e \delta \]
\[ \delta = \frac{PL}{AE} \]
\[ P = \left( \frac{AE}{L} \right) \delta \]
\[ k_e = \frac{AE}{L} \]
\[ A = \frac{\pi d^2}{4} = \frac{\pi (8 \times 10^{-3} \text{ m})^2}{4} \]
\[ A = 5.027 \times 10^{-5} \text{ m}^2 \]
\[ L = 0.450 \text{ m} \]
\[ E = 200 \times 10^9 \text{ N/m}^2 \]
\[ k_e = \frac{(5.027 \times 10^{-5} \text{ m}^2)(200 \times 10^9 \text{ N/m}^2)}{(0.450 \text{ m})} \]
\[ k_e = 22.34 \times 10^6 \text{ N/m} \quad k_e = 22.3 \text{ MN/m} \]

\((b)\)

\[ f_n = \frac{\sqrt{\frac{k_e}{m}}}{2\pi} \]
\[ = \frac{\sqrt{\frac{22.3 \times 10^6}{8}}}{2\pi} \]
\[ = 265.96 \text{ Hz} \quad f_n = 266 \text{ Hz} \]
PROBLEM 19.28

From mechanics of materials it is known that for a cantilever beam of constant cross section, a static load $P$ applied at end $B$ will cause a deflection $\delta_B = \frac{PL^3}{3EI}$, where $L$ is the length of the beam, $E$ is the modulus of elasticity, and $I$ is the moment of inertia of the cross-sectional area of the beam. Knowing that $L = 3$ m, $E = 230$ GPa, and $I = 5 \times 10^6$ mm$^4$, determine $(a)$ the equivalent spring constant of the beam, $(b)$ the frequency of vibration of a 250 kg block attached to end $B$ of the same beam.

SOLUTION

(a) Equivalent spring constant.

$$k_e = \frac{P}{\delta_B}$$

$$P = k_e \delta_B$$

$$\delta_B = \frac{PL^3}{3EI}$$

$$P = \left(\frac{3EI}{L^3}\right) \delta_B$$

$$k_e = \frac{3EI}{L^3}$$

$$= \frac{(3)(230 \times 10^9 \text{ N/m}^2)(5 \times 10^6 \times 10^{-12} \text{ m}^4)}{(3)^3}$$

$$k_e = 127.778 \times 10^3 \text{ N/m}$$

$k_e = 127.8 \text{ kN/m}$

(b) Natural frequency.

$$f_n = \frac{\sqrt{k_e}}{2\pi}$$

$$k_e = 127.778 \times 10^3 \text{ N/m}$$

$$f_n = \sqrt{\frac{(127.778 \times 10^3 \text{ N/m})}{250 \text{ kg}}}$$

$$f_n = 3.5981 \text{ Hz}$$

$f_n = 3.60 \text{ Hz}$
PROBLEM 19.29

A 40-mm deflection of the second floor of a building is measured directly under a newly installed 4000-kg piece of rotating machinery, which has a slightly unbalanced rotor. Assuming that the deflection of the floor is proportional to the load it supports, determine (a) the equivalent spring constant of the floor system, (b) the speed in rpm of the rotating machinery that should be avoided if it is not to coincide with the natural frequency of the floor-machinery system.

SOLUTION

(a) Equivalent spring constant.

\[ W = k_e \delta_s \]
\[ k_e = \frac{W}{\delta} = \frac{(4000)(9.81)}{(0.04 \text{ m})} = 981 \times 10^3 \text{ N/m} \]

\[ k_e = 981 \text{ kN/m} \]

(b) Natural frequency.

\[ f_n = \frac{\sqrt{k_e}}{2\pi} \]
\[ f_n = \frac{\sqrt{(981 \times 10^3 \text{ N/m})}}{2\pi} \]
\[ f_n = 2.4924 \text{ Hz} \]

1 Hz = 1 cycle/s

\[ = 60 \text{ rpm} \]

\[ \text{Speed} = (2.4924 \text{ Hz}) \frac{(60 \text{ rpm})}{\text{Hz}} \]
\[ = 149.5 \text{ rpm} \]
PROBLEM 19.30

The force-deflection equation for a nonlinear spring fixed at one end is \( F = 5x^{1/2} \) where \( F \) is the force, expressed in newtons, applied at the other end and \( x \) is the deflection expressed in meters. (a) Determine the deflection \( x_0 \) if a 120-g block is suspended from the spring and is at rest. (b) Assuming that the slope of the force-deflection curve at the point corresponding to this loading can be used as an equivalent spring constant, determine the frequency of vibration of the block if it is given a very small downward displacement from its equilibrium position and released.

SOLUTION

(a) Deflection \( x_0 \).
\[
mg = (0.120 \text{ kg})(9.81 \text{ m/s}^2)
\]
\[
mg = 1.177 \text{ N}
\]
\[
F = mg = 5x_0^{1/2}
\]
\[
x_0 = \left(\frac{1.177}{5}\right)^2
\]
\[
x_0 = 0.0554 \text{ m}
\]

Equivalent spring constant.
At \( x_0 \),
\[
\left(\frac{dF}{dx}\right)_{x_0} = \frac{5}{2}(x_0)^{-1/2} = \frac{5}{2}(0.0554)^{-1/2}
\]
\[
\left(\frac{dF}{dx}\right)_{x_0} = 10.618 \text{ N/m}
\]
\[
k_e = 10.618 \text{ N/m}
\]

(b) Natural frequency.
\[
f_n = \frac{\sqrt{\frac{k_e}{m}}}{2\pi}
\]
\[
f_n = \sqrt{\frac{(10.618 \text{ N/m})}{(0.120 \text{ kg})}}
\]
\[
f_n = 1.4971 \text{ Hz}
\]
PROBLEM 19.31

If $h = 700$ mm and $d = 500$ mm and each spring has a constant $k = 600$ N/m, determine the mass $m$ for which the period of small oscillations is (a) $0.50$ s, (b) infinite. Neglect the mass of the rod and assume that each spring can act in either tension or compression.

SOLUTION

\[
x_x = \frac{d}{h}
\]

\[
2F = 2kx_x = 2k\frac{d}{h}x
\]

\[
\sum C_{MA} = \sum (M_A)_{eff}: \quad 2Fd - mgx = -(m\ddot{x})h
\]

\[
2k\left(\frac{d}{h}x\right)d - mgx = -m\ddot{x}h
\]

\[
\ddot{x} + \left[\frac{2kd^2}{mh^2} - \frac{g}{h}\right]x = 0
\]

\[
\omega_n^2 = \left[\frac{2kd^2}{mh^2} - \frac{g}{h}\right]
\]

\[
\omega_n^2 = \frac{2k}{m}\left(\frac{d}{h}\right)^2 - \frac{g}{h}
\]

Data:

\[
d = 0.5 \text{ m}
\]

\[
h = 0.7 \text{ m}
\]

\[
k = 600 \text{ N/m}
\]
PROBLEM 19.31 (Continued)

(a) For $t = 0.5$ s:

$$t = \frac{2\pi}{\omega_n}; \quad 0.5 = \frac{2\pi}{\omega_n} \quad \omega_n = 4\pi$$

Eq. (1):

$$(4\pi)^2 = \frac{2(600)(0.5)^2}{m} - \frac{9.81}{0.7}$$

$$m = 3.561 \text{ kg}$$

(b) For $t$ infinite:

$$t = \frac{2\pi}{\omega_n} \quad \omega_n = 0$$

Eq. (1):

$$0 = \frac{2(600)(0.5)^2}{m} - \frac{9.81}{0.7}$$

$$m = 43.69 \text{ kg}$$
**PROBLEM 19.32**

Denoting by $\delta_{ST}$ the static deflection of a beam under a given load, show that the frequency of vibration of the load is

$$f = \frac{1}{2\pi} \sqrt{\frac{g}{\delta_{ST}}}$$

Neglect the mass of the beam, and assume that the load remains in contact with the beam.

**SOLUTION**

$$k = \frac{W}{\delta_{ST}}$$

$$m = \frac{W}{g}$$

$$\omega_n^2 = \frac{k}{m} = \frac{W}{\delta_{ST}} \frac{g}{W} = \frac{g}{\delta_{ST}}$$

$$F_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{g}{\delta_{ST}}}$$
PROBLEM 19.33*

Expanding the integrand in Equation (19.19) of Section 19.4 into a series of even powers of \( \sin \phi \) and integrating, show that the period of a simple pendulum of length \( l \) may be approximated by the formula

\[
\tau = 2\pi \sqrt{\frac{I}{g}} \left( 1 + \frac{1}{4} \sin^2 \frac{\theta_m}{2} \right)
\]

where \( \theta_m \) is the amplitude of the oscillations.

SOLUTION

Using the Binomial Theorem, we write

\[
\frac{1}{\sqrt{1 - \sin^2 \left( \frac{\theta_m}{2} \right)}} = \left[ 1 - \sin^2 \left( \frac{\theta_m}{2} \right) \sin \phi \right]^{-1/2}
\]

\[
= 1 + \frac{1}{2} \sin^2 \frac{\theta_m}{2} \sin^2 \phi + \cdots
\]

Neglecting terms of order higher than 2 and setting \( \sin^2 \phi = \frac{1}{2} (1 - \cos 2\phi) \), we have

\[
\tau_n = 4 \sqrt{\frac{I}{g}} \int_0^{2\pi} \left\{ 1 + \frac{1}{2} \sin^2 \frac{\theta_m}{2} \left[ \frac{1}{2} (1 - \cos 2\phi) \right] \right\} d\phi
\]

\[
= 4 \sqrt{\frac{I}{g}} \int_0^{2\pi} \left\{ 1 + \frac{1}{4} \sin^2 \frac{\theta_m}{2} - \frac{1}{4} \sin^2 \frac{\theta_m}{2} \cos 2\phi \right\} d\phi
\]

\[
= 4 \sqrt{\frac{I}{g}} \left[ \phi + \frac{1}{4} \left( \sin^2 \frac{\theta_m}{2} \right) \phi - \frac{1}{8} \sin^2 \frac{\theta_m}{2} \sin 2\phi \right]_0^{\pi/2}
\]

\[
= 4 \sqrt{\frac{I}{g}} \left[ \frac{\pi}{2} + \frac{1}{4} \left( \sin^2 \frac{\theta_m}{2} \right) \frac{\pi}{2} + 0 \right]
\]

\[
\tau_n = 2\pi \sqrt{\frac{I}{g}} \left( 1 + \frac{1}{4} \sin^2 \frac{\theta_m}{2} \right)
\]
PROBLEM 19.34*

Using the formula given in Problem 19.33, determine the amplitude $\theta_m$ for which the period of a simple pendulum is $\frac{1}{2}$ percent longer than the period of the same pendulum for small oscillations.

SOLUTION

For small oscillations,

$$\tau_n = 2\pi \sqrt{\frac{l}{g}}$$

We want

$$\tau_n = 1.005(\tau_n)_0$$

$$= 1.005 2\pi \sqrt{\frac{l}{g}}$$

Using the formula of Problem 19.33, we write

$$\tau_n = (\tau_n)_0 \left(1 + \frac{1}{4} \sin^2 \frac{\theta_m}{2}\right)$$

$$= 1.005(\tau_n)_0$$

$$\sin^2 \frac{\theta_m}{2} = 4[1.005 - 1] = 0.02$$

$$\sin \frac{\theta_m}{2} = \sqrt{0.02}$$

$$\frac{\theta_m}{2} = 8.130^\circ$$

$$\theta_m = 16.26^\circ$$
PROBLEM 19.35*

Using the data of Table 19.1, determine the period of a simple pendulum of length $l = 750$ mm (a) for small oscillations, (b) for oscillations of amplitude $\theta_m = 60^\circ$, (c) for oscillations of amplitude $\theta_m = 90^\circ$.

SOLUTION

(a) $\tau_n = 2\pi \sqrt{\frac{l}{g}}$ (Equation 19.18 for small oscillations):

\[
\tau_n = 2\pi \sqrt{\frac{0.750 \text{ m}}{9.81 \text{ m/s}^2}}
= 1.737 \text{ s}
\]

(b) For large oscillations (Eq. 19.20),

\[
\tau_n = \left(\frac{2k}{\pi}\right) \left(\frac{2\pi}{\sqrt{\frac{l}{g}}}\right)
= \frac{2k}{\pi} (1.737 \text{ s})
\]

For $\theta_m = 60^\circ$, $k = 1.686$ (Table 19.1)

\[
\tau_n(60^\circ) = \frac{2(1.686)(1.737 \text{ s})}{\pi}
= 1.864 \text{ s}
\]

(c) For $\theta_m = 90^\circ$, $k = 1.854$

\[
\tau_n = \frac{2(1.854)(1.737 \text{ s})}{\pi}
= 2.05 \text{ s}
\]
PROBLEM 19.36*

Using the data of Table 19.1, determine the length in mm of a simple pendulum which oscillates with a period of 2 s and an amplitude of 90°.

SOLUTION

For large oscillations (Eq. 19.20),

\[ \tau_n = \left( \frac{2k}{\pi} \right) \left( 2\pi \sqrt{\frac{l}{g}} \right) \]

for \( \theta_m = 90° \)

\[ k = 1.854 \text{ (Table 19.1)} \]

\[ (2 \text{ s}) = (2)(1.854)(2) \sqrt{\frac{l}{9.81 \text{ m/s}^2}} \]

\[ l = (2 \text{ s})^2 (9.81 \text{ m/s}^2) \]

\[ = 0.71349 \text{ m} \]

\[ l = 713 \text{ mm} \]
PROBLEM 19.37

The 5-kg uniform rod $AC$ is attached to springs of constant $k = 500 \text{ N/m}$ at $B$ and $k = 620 \text{ N/m}$ at $C$, which can act in tension or compression. If the end $C$ of the rod is depressed slightly and released, determine (a) the frequency of vibration, (b) the amplitude of the motion of Point $C$, knowing that the maximum velocity of that point is 0.9 m/s.

SOLUTION

Equation of motion,

$$\overrightarrow{(\Sigma M_A)}_{\text{eff}} = (\Sigma M_A) = \begin{align*} & (0.7)[k_B((0.7\theta + (\delta_{ST})_B) - mg] + 1.4[k_C(1.4\theta + (\delta_{ST})_C)] = -\overline{T} \alpha - (0.7)(ma_t) \\ & (1) \end{align*}$$

But in equilibrium $(\theta = 0)$,

$$\sqrt{+} \Sigma M_A = 0 = 0.7[k_B(\delta_{ST})_B - mg] + 1.4 k_C(\delta_{ST})_C$$

Substituting Equation (2) into Equation (1),

$$\overline{T} \alpha + 0.7ma_t + (0.7)^2 k_B \theta + (1.4)^2 k_C \theta = 0$$

Kinematics $(\alpha = \dot{\theta})$:

$$a_t = 0.7\alpha = 0.7\dot{\theta}$$

$$[\overline{T} + m(0.7)^2] \ddot{\theta} + [(0.7)^2 k_B + (1.4)^2 k_C] \theta = 0$$

$$\overline{T} = \frac{1}{12} ml^2$$

$$= \frac{1}{12}(5 \text{ kg})(1.4 \text{ m})^2$$

$$= 0.8167 \text{ kg} \cdot \text{m}^2$$
PROBLEM 19.37 (Continued)

\[(.7)^2 m = (0.49 \text{ m}^2)(5 \text{ kg})\]
\[= 2.45 \text{ kg} \cdot \text{m}^2\]
\[(0.7)^2 k_B + (1.4)^2 k_C = (0.49 \text{ m}^2)(500 \text{ N/m}) + (1.96 \text{ m}^2)(620 \text{ N/m})\]
\[= 245 + 1215.2\]
\[= 1460.2 \text{ N} \cdot \text{m}\]

\[[0.8167 + 2.45] \dot{\theta} + 1460.2 \theta = 0\]
\[\ddot{\theta} + \frac{(1460.2 \text{ N} \cdot \text{m})}{(3.267 \text{ kg} \cdot \text{m}^2)} \theta = 0\]
\[\ddot{\theta} + 447 \theta = 0 \quad (\text{N/kg} \cdot \text{m} = \text{s}^{-2})\]

(a) Natural frequency.
\[\omega_n = \sqrt{447 \text{ s}^{-2}}\]
\[= 21.14 \text{ rad/s}\]
\[f_n = \frac{\omega_n}{2\pi} = 21.14 \frac{\text{rad}}{2\pi} = 3.36 \text{ Hz}\]

(b) Maximum velocity.
\[\theta = \theta_m \sin(\omega_n t + \phi)\]
\[\dot{\theta} = (\theta_m)(\omega_n)\cos(\omega_n t + \phi)\]

Maximum angular velocity.
\[\dot{\theta}_m = \theta_m \omega_n\]

Maximum velocity at C.
\[(\dot{x}_C)_m = 1.4 \dot{\theta}_m\]
\[= (1.4 \text{ m})(\theta_m)\omega_n\]
\[\theta_m = \frac{0.9 \text{ m/s}}{(1.4 \text{ m})(21.14 \text{ rad/s})}\]
\[= 0.03041 \text{ rad}\]

Maximum amplitude at C.
\[(x_C)_m = (1.4 \text{ m})(\theta_m)\]
\[= (1.4 \text{ m})(0.03041)\]
\[(x_C)_m = 0.0426 \text{ m} \quad (x_C)_m = 42.6 \text{ mm}\]
PROBLEM 19.38

The uniform rod shown weighs 7.5 kg and is attached to a spring of constant \( k = 800 \text{ N/m} \). If end \( B \) of the rod is depressed 10 mm and released, determine (a) the period of vibration, (b) the maximum velocity of end \( B \).

SOLUTION

\[
k = 800 \text{ N/m} \\
m = 7.5 \text{ kg}
\]

where

\[
F = k(x + \delta_{ST}) = k(0.5\theta + \delta_{ST})
\]

\[
m\ddot{x} = m\ddot{r}\alpha = (7.5)(0.125 \text{ m})\ddot{\theta} = \frac{15}{16}\ddot{\theta}
\]

\[
\dot{T}\alpha = \frac{1}{12}(7.5)(0.75)^2\dot{\theta}
\]

\[
= \frac{45}{128}\dot{\theta}
\]

(a) Equation of motion.

\[
\sum M_C = \Sigma (M_C)_{\text{eff}}: \quad mg(0.125 \text{ m}) - F(0.5 \text{ m}) = \dot{T}\alpha + m\ddot{x}(0.125 \text{ m})
\]

\[
mg(0.125) - k(0.5\theta + \delta_{ST})(0.5 \text{ m}) = \frac{45}{128}\ddot{\theta} + \frac{15}{16}\ddot{\theta}(0.25)
\]

But in equilibrium, we have

\[
\sum mg(0.125) - k\delta_{ST}(0.5) = 0
\]
PROBLEM 19.38 (Continued)

Thus,

\[- k(0.5)^2 \theta = \frac{45}{128} + \frac{15}{128} \dot{\theta}\]

\[-(800)(0.5)^2 \theta = \frac{15}{32} \ddot{\theta}\]

\[\ddot{\theta} + (426.667)\theta = 0\]

Natural frequency and period.

\[\omega_n^2 = 426.667\]

\[\omega_n = 20.656 \text{ rad/s}\]

\[\tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{20.656 \text{ rad/s}}\]

\[\tau = 0.304 \text{ s} \uparrow\]

(b) At end \(B\).

\[x_m = 10 \text{ mm} = 0.01 \text{ m}\]

\[v_m = x_m \omega_n\]

\[= (0.01 \text{ m})(20.656 \text{ rad/s})\]

\[= 0.20656 \text{ m/s}\]

\[v_m = 0.207 \text{ m/s} \uparrow\]
**PROBLEM 19.39**

A 15 kg uniform cylinder can roll without sliding on a 15° incline. A belt is attached to the rim of the cylinder, and a spring holds the cylinder at rest in the position shown. If the center of the cylinder is moved 50 mm down the incline and released, determine (a) the period of vibration, (b) the maximum acceleration of the center of the cylinder.

**SOLUTION**

Spring deflection.

\[ x_d = x_0 + x_{d0} \]

\[ x_{d0} = r \theta \]

\[ \theta = \frac{x_0}{r} \]

\[ x_d = 2x_0 \]

\[ F_s = k(x_d + \delta_{ST}) = k(2x_0 + \delta_{ST}) \]

\[ \sum M_C = (\sum M)_{eff} : -2rk(2x_0 + \delta_{ST}) + rw \sin 15^\circ = rm\dddot{x}_0 + \ddot{T} \theta \]  

(1)

But in equilibrium,

\[ x_0 = 0 \]

\[ \sum M_C = 0 = -2rk\delta_{ST} + rw \sin 15^\circ \]  

(2)

Substituting Eq. (2) into Eq. (1) and noting that \( \theta = \frac{x_0}{r}, \quad \ddot{\theta} = \frac{\dddot{x}_0}{r} \)

\[ rm\dddot{x}_0 + T \frac{\dddot{x}_0}{r} + 4rkx_0 = 0 \]

\[ T = \frac{1}{2}mr^2 \]

\[ \frac{3}{2} mx_0 + 4rkx_0 = 0 \]

\[ \dddot{x}_0 + \left( \frac{8 k}{3 m} \right) x_0 = 0 \]
PROBLEM 19.39 (Continued)

Natural frequency. 
\[ \omega_n = \sqrt{\frac{8 \cdot k}{3 \cdot m}} = \sqrt{\frac{(8)(6000 \, \text{N/m})}{(3)(15 \, \text{kg})}} = 32.6599 \, \text{s}^{-1} \]

(a) Period. 
\[ \tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{32.6599} \]
\[ \tau_n = 0.1924 \, \text{s} \]

(b) 
At \( t = 0, \)
\[ x_0 = (x_0)_m \sin(\omega_n t + \phi) \]
\[ x_0 = 0.05 \, \text{m} \quad \dot{x}_0 = 0 \]
\[ \ddot{x}_0 = (x_0)_m \omega_n \cos(\omega_n t + \phi) \]
\[ t = 0 \]
\[ 0 = (x_0)_m \omega_n \cos \phi \]
Thus,
\[ \phi = \frac{\pi}{2} \]
\[ t = 0 \]
\[ x_0(0) = 0.05 \, \text{m} = (x_0)_m \sin \phi = (x_0)_m (1) \]
\[ (x_0)_m = 0.05 \, \text{m} \]
\[ \dddot{x}_0 = -(x_0)_m \omega_n^2 \sin(\omega_n t + \phi) \]
\[ (a_0)_{\text{max}} = (\dddot{x}_0)_{\text{max}} \]
\[ = -(x_0)_m \omega_n^2 \]
\[ = -(0.05 \, \text{m})(32.6599 \, \text{s}^{-1})^2 \]
\[ = 53.333 \, \text{m/s}^2 \]
\[ (a_0)_{\text{max}} = 53.3 \, \text{m/s}^2 \]
PROBLEM 19.40

A 7.5 kg slender rod $AB$ is riveted to a 6 kg uniform disk as shown. A belt is attached to the rim of the disk and to a spring which holds the rod at rest in the position shown. If end $A$ of the rod is moved 20 mm down and released, determine (a) the period of vibration, (b) the maximum velocity of end $A$.

SOLUTION

Equation of motion. 

\[ \Sigma M_B = \Sigma (M_B)_{\text{eff}}: \quad mgL/2 \cos \theta - kx = I_{AB}\alpha + m(\alpha L/2)(L/2) + I_{\text{Disk}}\alpha \]  

where $x = r\theta + \delta_{ST}$ and from statics, $mgL/2 = k\delta_{ST}r$

Assuming small angles ($\cos \theta = 1$), Equation (1) becomes

\[ mgL/2\theta - kr^2\theta - k\delta_{ST} = \left( I_{AB} + m(L/2)^2 + I_{\text{Disk}} \right)\alpha \]

\[ \left( I_{AB} + mL^2/4 + I_{\text{Disk}} \right)\dot{\theta} + kr^2\theta = 0 \]

Data:

- $m = 7.5$ kg
- $m_{\text{disk}} = 6$ kg
- $L = 900$ mm $= 0.9$ m
- $r = 250$ mm $= 0.25$ m
- $k = 6$ kN/m $= 6000$ N/m
PROBLEM 19.40 (Continued)

\[ I_{AB} = \frac{1}{12} mL^2 \]
\[ = \frac{1}{12} (7.5)(0.9)^2 \]
\[ = 0.50625 \text{ kg} \cdot \text{m}^2 \]

\[ I_{\text{Disk}} = \frac{1}{2} m_{\text{Disk}} r^2 \]
\[ = \frac{1}{2} (6)(0.25)^2 \]
\[ = 0.1875 \text{ kg} \cdot \text{m}^2 \]

\[ \left[ 0.50625 + \frac{1}{4} (7.5)(0.9)^2 + 0.1875 \right] \dot{\theta} + (6000)(0.25)^2 \theta = 0 \]
\[ 2.2125 \dot{\theta} + 375 \theta = 0 \text{ or } \dot{\theta} + 169.492 \theta = 0 \]

(a) Natural frequency and period.
\[ \omega_n^2 = 169.492 \text{ (rad/s)}^2 \]
\[ \omega_n = 13.0189 \text{ rad/s} \]
\[ \tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{13.0189} \quad \tau = 0.483 \text{ s} \]

(b) Maximum velocity.
\[ v_m = \omega_n x_m = (13.0189 \text{ rad/s})(0.02 \text{ m}) \quad v_m = 260 \text{ m/s} \]
PROBLEM 19.41

An 8-kg uniform rod \( AB \) is hinged to a fixed support at \( A \) and is attached by means of pins \( B \) and \( C \) to a 12-kg disk of radius 400 mm. A spring attached at \( D \) holds the rod at rest in the position shown. If Point \( B \) is moved down 25 mm and released, determine (a) the period of vibration, (b) the maximum velocity of Point \( B \).

SOLUTION

(a)

Equation of motion.

\[
\sum M_A = (\sum M_A)_{\text{eff}}: \quad F_S = k(0.6\theta + \delta_{ST}) \\
\Rightarrow 0.6(m_R g - F_S) + 1.2m_D g = (\ddot{T}_R + \ddot{T}_D)\alpha + 0.6(m_R)(a_B) + 1.2(m_D)(a_B) \\
\]

At equilibrium \((\theta = 0)\),

\[
F_S = k\delta_{ST} \\
\sum M_A = 0 = 0.6(m_R g - k(\delta_{ST})) + 1.2m_D g \\
\]

Substituting Eq. (2) into Eq. (1),

\[
(\ddot{T}_R + \ddot{T}_D)\alpha + 0.6 m_R(a_B) + 1.2m_D(a_B) + (0.6)^2 k\theta = 0 \\
\alpha = \ddot{\theta} \\
(a_B) = 0.6\ddot{\theta} \\
(a_B) = 1.2\ddot{\theta} \\
\ddot{T}_R = \frac{1}{12}m_R\dot{\theta}^2 = \frac{1}{12}(8)(1.2)^2 \\
= 0.960 \text{ kg} \cdot \text{m}
PROBLEM 19.41 (Continued)

\[
\begin{aligned}
T_D &= \frac{1}{2} m_D R^2 = \frac{1}{2} (12)(0.4)^2 = 0.960 \text{ kg} \cdot \text{m} \\
[0.960 + 0.960 + (0.6)^2 (8) + (1.2)^2 (12)] \ddot{\theta} + (0.6)^2 (800) \theta &= 0 \\
\ddot{\theta} + \frac{288 \text{ N} \cdot \text{m}}{22.08 \text{ kg} \cdot \text{m}^2} \theta &= 0
\end{aligned}
\]

(a) Natural frequency and period.

\[
\begin{aligned}
\omega_n &= \sqrt{\frac{288}{22.08}} \\
&= 3.6116 \text{ rad/s} \\
\tau_n &= \frac{2\pi}{\omega_n} = \frac{2\pi}{3.6116} \\
&= 1.740 \text{ s} \quad \blacktriangle
\end{aligned}
\]

(b) Maximum velocity at \(B\).

\[
\begin{aligned}
(v_B)_{\text{max}} &= (1.2)(\dot{\theta})_{\text{max}} \\
\dot{\theta} &= \frac{v_B}{1.2} \\
\theta &= \frac{0.025}{1.2} = 0.02083 \text{ rad} \\
\dot{\theta} &= \theta_m \omega_n \sin(\omega_n t + \phi) \\
(\dot{\theta})_{\text{max}} &= \frac{\theta_m \omega_n}{1.2} = (0.02083)(3.612) = 0.07524 \text{ rad/s} \\
(v_B)_{\text{max}} &= (1.2)(\dot{\theta})_{\text{max}} = (1.2 \text{ m})(0.07524) \text{ rad/s} \quad \blacktriangledown \\
(\dot{v}_B)_{\text{max}} &= 0.09029 \text{ m/s} \\
(\dot{v}_B)_{\text{max}} &= 90.3 \text{ mm/s} \quad \blacktriangle
\end{aligned}
\]
PROBLEM 19.42

Solve Problem 19.41, assuming that pin \( C \) is removed and that the disk can rotate freely about pin \( B \).

SOLUTION

(a)

Note: This problem is the same as Problem 19.41, except that the disk does not rotate, so that the effective moment \( I_D \alpha = 0 \).

Equation of motion,

\[
\Sigma M_A = (\Sigma M_A)_{\text{eff}}: \quad F_S = k(0.60 + \delta_{ST}) \]

\[
(0.6)(m_Rg - F_S) + 1.2m_Dg = I_R\alpha + (0.6)(m_R)(a_t)_D + 1.2(m_D)(a_t)_B \quad (1)
\]

At equilibrium \( (\theta = 0) \),

\[
F_S = k\delta_{ST} \]

\[
\Sigma M_A = 0 = 0.6(m_Rg - \delta_{ST}) + 1.2m_Dg \quad (2)
\]

Substituting Eq. (2) into Eq. (1),

\[
I_R\alpha + 0.6 m_R(a_t)_D + 1.2m_D(a_t)_B + (0.6)^2 k\theta = 0
\]

\[
\alpha = \ddot{\theta}
\]

\[
(a_t)_B = 0.6\ddot{\theta}
\]

\[
(a_t)_D = 1.2\ddot{\theta}
\]

\[
I_R = \frac{1}{12} m_R l^2 = \frac{1}{12} (8)(1.2)^2
\]

\[
= 0.960 \text{ kg} \cdot \text{m}
\]
PROBLEM 19.42 (Continued)

\[ [0.960 + (0.6)^2(8) + (1.2)^2(12)]\ddot{\theta} + (0.6)^2(800)\theta = 0 \]
\[ \ddot{\theta} + \frac{(288 \text{ N \cdot m})}{21.12 \text{ kg \cdot m}^2} \theta = 0 \]

(a) Natural frequency and period.

\[ \omega_n = \sqrt{\frac{288}{21.12}} \]
\[ = 3.693 \text{ rad/s} \]
\[ \tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{3.693} \]
\[ \tau_n = 1.701 \text{ s} \]

(b) Maximum velocity at B.

\[ (v_B)_{\text{max}} = (1.2)(\dot{\theta})_{\text{max}} \]
\[ \dot{\theta}_m = \frac{\mu_B}{1.2} = \frac{0.025}{1.20} = 0.02083 \text{ rad} \]
\[ \theta = \theta_m \sin(\omega_n t + \phi) \]
\[ \dot{\theta} = \theta_m \omega_n \cos(\omega_n t + \phi) \]
\[ \ddot{\theta}_{\text{max}} = \theta_m \omega_n \]
\[ = (0.02083)(3.693) \]
\[ = 0.07694 \text{ rad/s} \]
\[ (v_B)_{\text{max}} = (1.2)(\dot{\theta}_{\text{max}}) \]
\[ = (1.2)(0.07694) \]
\[ = 0.09233 \text{ m/s} \]
\[ (v_B)_{\text{max}} = 92.3 \text{ mm/s} \]
PROBLEM 19.43

A belt is placed around the rim of a 240-kg flywheel and attached as shown to two springs, each of constant \( k = 15 \text{ kN/m} \). If end \( C \) of the belt is pulled 40 mm down and released, the period of vibration of the flywheel is observed to be 0.5 s. Knowing that the initial tension in the belt is sufficient to prevent slipping, determine (a) the maximum angular velocity of the flywheel, (b) the centroidal radius of gyration of the flywheel.

SOLUTION

Denote the initial tension by \( T_0 \).

Equation of motion.

\[
\sum M_0 = \Sigma (M_0)_{eff}: \quad -T_A r + T_B r = \dot{T} \dot{\theta} \\
- (T_0 + k r \theta) r + (T_0 - k r \theta) \theta = \dot{T} \dot{\theta} \\
\dot{\theta} + \frac{2k r^2}{T} \dot{\theta} = 0 \\
\omega_n = \frac{2k r^2}{T} 
\]

Data:

\( m = 240 \text{ kg} \) \quad \( k = 15 \text{ kN/m} \)
\( \tau = 0.5 \text{ s} \) \quad \( r = 0.45 \text{ m} \)
\[ \tau = \frac{2\pi}{\omega_n}; \quad \omega_n = \frac{2\pi}{\tau} = \frac{2\pi}{0.5} = 4\pi \text{ rad/s} \]

(a) Maximum angular velocity. If Point \( C \) is pulled down 40 mm and released,

\[
\theta_m = \theta_{max} \\
= \left( \frac{40 \text{ mm}}{450 \text{ mm}} \right) \\
= 88.89 \times 10^{-3} \text{ rad} \\
\dot{\theta}_m = \theta_m \omega_n \\
= (88.92 \times 10^{-3} \text{ rad})(4\pi \text{ rad/s}) \quad \dot{\theta}_m = 1.117 \text{ rad/s} \]

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PROBLEM 19.43 (Continued)

(b) Centroidal radius of gyration.

\[ \omega_n^2 = \frac{2kr^2}{I} \]

\[(4\pi \text{ rad/s})^2 = \frac{2(15 \text{ kN/m})(0.45 \text{ m})^2}{I} \]

or since

\[ I = 38.47 \text{ kg} \cdot \text{m}^2 \]

\[ I = mk^2 \]

\[(240 \text{ kg})\tilde{k}^2 = 28.47 \text{ kg} \cdot \text{m}^2 \]

\[ \tilde{k} = 0.4004 \text{ m} \]

\[ \tilde{k} = 400 \text{ mm} \]
PROBLEM 19.44

A 75-mm-radius hole is cut in a 200-mm-radius uniform disk, which is attached to a frictionless pin at its geometric center \( O \). Determine (a) the period of small oscillations of the disk, (b) the length of a simple pendulum which has the same period.

SOLUTION

Equation of motion,

\[
\Sigma M_0 = (\Sigma M_0)_{\text{eff}}: \quad -m_H g(0.1) \sin \theta = I_D \ddot{\theta} - I_H \dddot{\theta} - (0.1)^2 m_H \dddot{\theta}
\]

\[
m_D = \rho t \pi R^2
\]

\[
= (\rho t \pi)(0.2)^2
\]

\[
= (0.04) \pi pt
\]

\[
m_H = \rho t \pi r^2
\]

\[
= (\rho t \pi)(0.075)^2
\]

\[
= (0.005625) \pi pt
\]

\[
I_D = \frac{1}{2} m_D R^2 = \frac{1}{2} (0.04 \pi pt)(0.2)^2
\]

\[
= 800 \times 10^{-6} \pi pt
\]

\[
I_H = \frac{1}{2} m_H r^2
\]

\[
= \frac{1}{2} (0.005625 \pi pt)(0.75)^2
\]

\[
= 15.82 \times 10^{-6} \pi pt
\]
PROBLEM 19.44 (Continued)

Small angles. \( \sin \theta = \theta \)

\[
\begin{align*}
[(800 \times 10^{-6} \pi - 15.82 \times 10^{-6} \pi - (0.1)^2 (0.005625 \pi)) &\rho f \ddot{\theta} \\
+ (0.005625) \pi \rho f (9.81)(0.1)\theta &= 0 \\
727.9 \times 10^{-6} \ddot{\theta} + 5.518 \times 10^{-3} \theta &= 0
\end{align*}
\]

(a) **Natural frequency and period.**

\[
\omega_n^2 = \frac{5.518 \times 10^{-3}}{727.9 \times 10^{-6}}
= 7.581
\]

\( \omega_n = 2.753 \text{ rad/s} \)

\[
\tau_n = \frac{2\pi}{\omega_n}
\]

\( \tau_n = \frac{2\pi}{2.753} = 2.28 \text{ s} \)

(b) **Length and period of a simple pendulum.**

\[
\tau_n = 2\pi \sqrt{\frac{l}{g}}
\]

\[
l = \frac{(\tau_n}{2\pi})^2 g
\]

\[
l = \left[ \frac{(2.753)}{2\pi} \right]^2 (9.81 \text{ m/s}^2)
\]

\( l = 1.294 \text{ m} \)
PROBLEM 19.45

Two small weights $w$ are attached at $A$ and $B$ to the rim of a uniform disk of radius $r$ and weight $W$. Denoting by $\tau_0$ the period of small oscillations when $\beta = 0$, determine the angle $\beta$ for which the period of small oscillations is $2\tau_0$.

SOLUTION

Equation of motion.

$$\Sigma M_c = (\Sigma M_c)_{\text{eff}}: \quad wr \sin(\beta - \theta) - wr \sin(\beta + \theta) - \frac{2w}{g} \alpha_t + I \alpha$$

$$wr \sin(\beta - \theta) - \sin(\beta + \theta) = -2wr \sin \theta \cos \beta$$

$$\sin \theta = \theta$$

$$\left( \frac{2w}{g} r^2 + \frac{W}{2g} r^2 \right) \ddot{\theta} + (2wr \cos \beta) \theta = 0$$

Natural frequency.

$$\omega_n = \sqrt{\frac{2wg \cos \beta}{2w + \frac{W}{r}}} r = \sqrt{\frac{4g \cos \beta}{4 + \frac{W}{r}}} r$$

$$\beta = 0 \quad \tau_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{\frac{4g}{(4 + \frac{W}{r})}}} r$$

$$\tau_n = \frac{2\pi}{\cos \beta \sqrt{\frac{4}{(4 + \frac{W}{r})}}} = 2\tau_0 = \frac{4\pi}{\sqrt{\frac{4g}{(4 + \frac{W}{r})}}} r$$

$$\cos \beta = \left( \frac{1}{2} \right)^2 = \frac{1}{4} \quad \beta = 75.5^\circ$$

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PROBLEM 19.46

Two 50 g weights are attached at A and B to the rim of a 1.5 kg uniform disk of radius \( r = 100 \text{ mm} \). Determine the frequency of small oscillations when \( \beta = 60^\circ \).

SOLUTION

\[ \alpha = \ddot{\theta} \]
\[ a_t = r\alpha = r\ddot{\theta} \]
\[ T_D = \frac{1}{2} \frac{W}{g} \]

Equation of motion.

\[ \Sigma M_C = (\Sigma M_C)_{\text{eff}}: \quad wr\sin(\beta - \theta) - wr\sin(\beta + \theta) = \frac{2w}{g} ra_t + T\alpha \]
\[ wr[\sin(\beta - \theta) - \sin(\beta + \theta)] = -2wr\sin\theta\cos\beta \]
\[ \sin\theta = \theta \]
\[ \left( \frac{2w}{g} r^2 + \frac{W}{2g} r^2 \right) \ddot{\theta} + (2wr\cos\beta)\dot{\theta} = 0 \]

Natural frequency.

\[ \omega_n = \frac{2wg \cos\beta}{\sqrt{(2w + \frac{W}{w})r}} = \frac{4g \cos\beta}{\sqrt{(4 + \frac{W}{w})r}} \]

Data:

\[ \frac{W}{w} = \frac{m}{m} = 1.5 \]
\[ m = 50 \times 10^{-3} = 30 \]
\[ r = 100 \text{ mm} = 0.1 \text{ m} \]
\[ \beta = 60^\circ \]
\[ \omega_n = \frac{(4)(9.81)\cos60^\circ}{\sqrt{(4 + 30)(0.1)}} = 2.4022 \text{ rad/s} \]
\[ f_n = \frac{\omega_n}{2\pi} = \frac{2.4022}{2\pi} \]

\[ f_n = 0.382 \text{ Hz} \]
PROBLEM 19.47

For the uniform square plate of side \( b = 300 \text{ mm} \), determine (a) the period of small oscillations if the plate is suspended as shown, (b) the distance \( c \) from \( O \) to a Point \( A \) from which the plate should be suspended for the period to be a minimum.

SOLUTION

(a) Equation of motion.

\[
\Sigma M_0 = (\Sigma M_0)_\text{eff}: \quad \alpha = \ddot{\theta} \\
\quad \bar{T} = \frac{1}{6} mb^2 \\
\quad a_r = (OG)(\alpha) \\
\quad OG = b \frac{\sqrt{2}}{2} \\
\quad a_r = \left( b \frac{\sqrt{2}}{2} \right) \ddot{\theta} \\
\Rightarrow (OG)(\sin \theta)(mg) = -(OG)ma_r - \bar{T} \alpha \quad \sin \theta = \theta \\
\quad \left( b \frac{\sqrt{2}}{2} \right) m \left( b \frac{\sqrt{2}}{2} \right) \ddot{\theta} + \frac{1}{6} mb^2 \dddot{\theta} + \left( b \frac{\sqrt{2}}{2} \right) mg \theta = 0 \\
\quad (b) \left( \frac{1}{2} + \frac{1}{6} \right) m \dddot{\theta} + \left( \frac{\sqrt{2}}{2} \right) mg \theta = 0
\]
PROBLEM 19.47 (Continued)

\[ \ddot{\theta} + \left( \frac{\sqrt{2}}{4} \right) g \theta = 0 \quad \text{or} \quad \ddot{\theta} + \frac{3\sqrt{2}}{4} g \theta = 0 \]

Natural frequency and period.

\[ \omega_n^2 = \frac{3\sqrt{2}}{4} \frac{g}{b} = \frac{(9.81)}{(0.3)} = 34.68 \]

\[ \omega_{n0} = 5.889 \text{ rad/s} \]

\[ \tau_{n0} = \frac{2\pi}{\omega_{n0}} \]

\[ \tau_{n0} = 1.067 \text{ s} \quad \blacktriangleleft \]

(b) Suspended about \( A \). Let \( e = (OG - c) \)

\[ a_\theta = e \alpha \]

Equation of motion.

\[ \sum M_A = \Sigma(M_A)_{\text{eff}} : \quad mge \sin \theta = -ema_\theta - \bar{T} \alpha = -(me^2 + \bar{T}) \alpha \]

\[ m\left(e^2 + \frac{1}{6}b^2\right)\dot{\theta} + mge\theta = 0 \]

Frequency and period.

\[ \omega_n^2 = \frac{eg}{e^2 + \frac{1}{6}b^2} \]

\[ \tau_n^2 = \frac{4\pi^2}{\omega_n^2} = \frac{4\pi^2(e^2 + \frac{1}{6}b^2)}{eg} \]

\[ \tau_n^2 = \frac{4\pi^2}{g} \left( e + \frac{b^2}{6e} \right) \]

For \( \tau_n \) to be minimum,

\[ \frac{d}{de} \left( e + \frac{b^2}{6e} \right) = 0 \]

\[ 1 - \frac{b^2}{6e^2} = 0 \quad \frac{b^2}{e^2} = 6 \quad e = \frac{b}{\sqrt{6}} \]

\[ c = OG - e = \frac{\sqrt{2}}{2} b - \frac{b}{\sqrt{6}} = 0.29886b \]

\[ c = (0.29886)(300 \text{ mm}) \quad c = 89.7 \text{ mm} \quad \blacktriangleleft \]
PROBLEM 19.48

A connecting rod is supported by a knife-edge at Point A; the period of its small oscillations is observed to be 0.87 s. The rod is then inverted and supported by a knife-edge at Point B and the period of small oscillations is observed to be 0.78 s. Knowing that \( r_a + r_b = 250 \text{ mm} \) determine (a) the location of the mass center \( G \), (b) the centroidal radius of gyration \( k \).

SOLUTION

Consider general pendulum of centroidal radius of gyration \( k \).

Equation of motion.

\[ -mg\bar{r} \sin \theta = (m\bar{r}\ddot{\theta})\bar{r} + mk^2 \ddot{\theta} \]

\[ \ddot{\theta} + \frac{g\bar{r}}{\bar{r}^2 + k^2} \sin \theta = 0 \]

For small oscillations, \( \sin \theta = \theta \), we have

\[ \ddot{\theta} + \frac{g\bar{r}}{\bar{r}^2 + k^2} \theta = 0 \]

\[ \omega_n^2 = \frac{g\bar{r}}{\bar{r}^2 + k^2} \]

\[ \tau = \frac{2\pi}{\omega_n} = 2\pi \sqrt{\frac{\bar{r}^2 + k^2}{g\bar{r}}} \]

For rod suspended at \( A \),

\[ R_A = 2\pi \sqrt{\frac{r_a^2 + k^2}{gr_a}} \]

\[ gR_A^2 = 4\pi^2 (r_a^2 + k^2) \quad (1) \]

For rod suspended at \( B \),

\[ R_B = 2\pi \sqrt{\frac{r_b^2 + k^2}{gr_b}} \]

\[ gR_B^2 = 4\pi^2 (r_b^2 + k^2) \quad (2) \]
PROBLEM 19.48 (Continued)

(a) Value of \( r_a \).

Subtracting Eq. (2) from Eq. (1),
\[
4\pi^2 (r_a^2 - r_b^2) = 4\pi^2 r_a^2 - 4\pi^2 r_b^2 = g r_a^2 t' - g r_b^2 t' = 4\pi^2 (r_a^2 - r_b^2)
\]

Applying the numerical data with \( r_a + r_b = 250 \text{ mm} = 0.25 \text{ m} \)
\[
(9.81)(0.87)^2 r_a^2 - (9.81)(0.78)^2 r_b^2 = 4\pi^2 (0.25)(r_a - r_b)
\]
\[
7.4252 r_a^2 - 5.9684 r_b = 9.8696 (r_a - r_b)
\]
\[
3.9012 r_b = 2.4444 r_a
\]
\[
r_b = 0.6266 r_a
\]
\[
0.25 = r_a + 0.6266 r_a
\]
\[
r_a = 0.15369 \text{ m}
\]
\[
r_b = 0.09631 \text{ m}
\]
\[
r_h = 96.31 \text{ mm} \]

(b) Centroidal radius of gyration.

From Eq. (1),
\[
4\pi^2 k^2 = 4\pi^2 r_a^2 - 4\pi^2 r_a^2
\]
\[
= (9.81)(0.87)^2 (0.15369) - 4\pi^2 (0.15369)^2 = 0.20867 \text{ m}^2
\]
\[
k = 0.072703 \text{ m}
\]
\[
k = 72.7 \text{ mm}
\]
PROBLEM 19.49

For the uniform equilateral triangular plate of side \( l = 300 \text{ mm} \), determine the period of small oscillations if the plate is suspended from (a) one of its vertices, (b) the midpoint of one of its sides.

SOLUTION

For an equilateral triangle,
\[
h = \frac{\sqrt{3}}{2} b, \quad A = \frac{1}{2} bh = \frac{\sqrt{3}}{4} b^2
\]

\[
T_x = \frac{1}{36} bh^3 = \frac{1}{36} \left( \frac{\sqrt{3}}{2} l \right)^3 = \frac{\sqrt{3}}{96} l^4
\]

\[
T_y = 2 - \frac{1}{12} h \left( \frac{b}{2} \right)^3 = \frac{1}{6} \left( \frac{\sqrt{3}}{2} l \right) \left( \frac{l}{2} \right)^3 = \frac{\sqrt{3}}{96} l^4
\]

\[
T_z = T_x + T_y = \frac{\sqrt{3}}{48} l^4
\]

\[
k^2 = \frac{T_z}{A} = \frac{1}{12} l^2 = \frac{1}{12} (0.300)^2 = 0.0075 \text{ m}^2
\]

Let \( r \) be the distance from the suspension point to the center of mass.

Equation of motion:
\[
\sum M_0 = \sum (M_0)_{\text{eff}}: \quad -mg \sin \theta = r (m \ddot{r} \dot{\theta}) + \ddot{r} \theta
\]

where
\[
T = mk^2
\]

For small angle \( \theta \),
\[
-mrg \theta = m (r^2 + k^2) \dot{\theta}
\]
\[
\ddot{\theta} + \frac{gr}{r^2 + k^2} \theta = 0
\]

Natural frequency:
\[
\omega_n^2 = \frac{gr}{r^2 + k^2}
\]
PROBLEM 19.49 (Continued)

(a) Plate is suspended from a vertex.
\[
r = \frac{2}{3} h = \frac{2}{3} \frac{\sqrt{3}}{2} l = \frac{2}{3} \frac{\sqrt{3}}{2} (0.3 \text{ m})
\]
\[
= 0.173205 \text{ m}
\]
\[
r^2 = 0.03 \text{ m}^2
\]
\[
\omega_n^2 = \frac{(9.81)(0.173205)}{0.03 + 0.0075} = 45.31
\]
\[
\omega_n = 6.7313 \text{ rad/s}
\]
\[
\tau = \frac{2\pi}{\omega_n}
\]
\[
\tau = 0.933 \text{ s} \quad \uparrow
\]

(b) Plate is suspended at the midpoint of one side.
\[
r = \frac{1}{3} h = \frac{1}{3} \left( \frac{\sqrt{3}}{2} l \right) = \frac{1}{3} \frac{\sqrt{3}}{2} (0.3)
\]
\[
= 0.0866025 \text{ m}
\]
\[
r^2 = 0.0075 \text{ m}^2
\]
\[
\omega_n^2 = \frac{(9.81)(0.0866025)}{0.0075 + 0.0075} = 56.64
\]
\[
\omega_n = 7.5258 \text{ rad/s}
\]
\[
\tau = \frac{2\pi}{\omega_n}
\]
\[
\tau = 0.835 \text{ s} \quad \uparrow
PROBLEM 19.50

A uniform disk of radius $r = 250$ mm is attached at $A$ to a 650-mm rod $AB$ of negligible mass, which can rotate freely in a vertical plane about $B$. Determine the period of small oscillations $(a)$ if the disk is free to rotate in a bearing at $A$, $(b)$ if the rod is riveted to the disk at $A$.

SOLUTION

$T = \frac{1}{2} mr^2$

$= \frac{1}{2} (0.250)^2 m = \frac{m}{32}$

$a_t = l \alpha = 0.650 \alpha$

$\alpha = \dot{\theta}$

$(a)$ The disk is free to rotate and is in **curvilinear translation**.

Thus, $T \alpha = 0$

$\Sigma M_B = \Sigma (M_B)_{eff}$:

$\sum (-mgl \sin \theta = lma_t)$

$\sin \theta = \theta$

$m l^2 \ddot{\theta} - mgl \theta = 0$

$\omega_n^2 = \frac{g}{l} = \frac{9.81 \text{ m/s}^2}{0.650 \text{ m}}$

$= 15.092$

$\omega_n = 3.885 \text{ rad/s}$

$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{3.885}$

$\tau_n = 1.617 \text{ s}$
PROBLEM 19.50 (Continued)

(b) When the disk is riveted at A, it rotates at an angular acceleration $\alpha$.

\[ \Sigma M_B = \Sigma (M_B)_{\text{eff}}: \]
\[ (-mgl \sin \theta) = I \alpha + lma \]
\[ I = \frac{1}{2} mr^2 \]
\[ \left( \frac{1}{2} mr^2 + ml^2 \right) \ddot{\theta} + mgl \theta = 0 \]

\[ \omega_n^2 = \frac{gl}{(l^2 + l^2)} \]
\[ = \frac{(9.81 \text{ m/s}^2)(0.650 \text{ m})}{\left( \frac{0.250^2}{2} \right) + (0.650)^2} \]
\[ = 14.053 \]
\[ \omega_n = 3.749 \text{ rad/s} \]
\[ \tau_n = \frac{2\pi}{\omega_n} \]
\[ = \frac{2\pi}{3.749} \]
\[ \tau_n = 1.676 \text{ s} \]
PROBLEM 19.51

A small collar weighing 1 kg is rigidly attached to a 3 kg uniform rod of length \( L = 1 \text{ m} \). Determine \((a)\) the distance \( d \) to maximize the frequency of oscillation when the rod is given a small initial displacement, \((b)\) the corresponding period of oscillation.

SOLUTION

Equation of motion.

\[
\Sigma M_A = (\Sigma M_A)_{\text{eff}}: -W_R \frac{L}{2} \sin \theta - W_C d \sin \theta = \ddot{\theta} R + \frac{L}{2} (\ddot{\alpha}_R) + m_c d (\ddot{\alpha}_C)
\]

\[
\sin \theta = \theta \quad \alpha = \dot{\theta}, \quad (\ddot{\alpha}_R) = \frac{L}{2} \ddot{\theta}, \quad (\ddot{\alpha}_C) = d \ddot{\theta}
\]

\[
\left( \ddot{\theta} + m_R \left( \frac{L}{2} \right)^2 + m_c d^2 \right) \ddot{\theta} + \left( m_R g \frac{L}{2} + m_c g d \right) \theta = 0
\]

\[
\ddot{\theta} = \frac{1}{12} m_R L^2
\]

\[
\ddot{\theta} + m_R \left( \frac{L}{2} \right)^2 = \frac{m_R L^2}{3}
\]

\[
\left( \frac{m_R L^2}{3} + m_c d^2 \right) \ddot{\theta} + \left( m_R g \frac{L}{2} + m_c g d \right) \theta = 0
\]
PROBLEM 19.51 (Continued)

\[ \ddot{\theta} + \left( \frac{\frac{1}{2} + \frac{m_c}{m_g} d}{\frac{1}{3} + \frac{m_c}{m_g} d^2} \right) g \theta = 0 \]
\[ \frac{m_c}{m_g} = \frac{1}{3} \quad L = 1 \text{ m} \]

\[ \ddot{\theta} + \left( \frac{\frac{1}{2} + \frac{1}{3} d}{\frac{1}{3} + \frac{1}{3} d^2} \right) \theta = 0 \]

Natural frequency.

\[ \omega_n^2 = \frac{\left( \frac{1}{2} + \frac{1}{3} d \right) g}{\frac{1}{3} + \frac{1}{3} d^2} = \frac{(1.5 + d) g}{(1 + d^2)} \]

(a) To maximize the frequency, we need to take the derivative and set \( A \) equal to zero.

\[ \frac{1}{g} \frac{d}{dt} \left( \omega_n^2 \right) = \frac{(1 + d^2)(1) - (1.5 + d)(2d)}{(1 + d^2)^2} = 0 \]

\[ d^2 + 3d - 1 = 0 \]

Solve for \( d \).

\[ d = 0.3028 \quad \text{or} \quad -3.3028 \]

\[ d = 0.303 \text{ m} \]

\[ \omega_n^2 = \frac{(1.5 + 0.3028)(9.81)}{(1 + 0.3028)^2} \]

\[ \omega_n^2 = 16.200 \]

\[ \omega_n = 4.0249 \text{ rad/s} \]
\[ \omega_n = 4.02 \text{ rad/s} \]

(b) Period of oscillation.

\[ \tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{4.0249} \]

\[ \tau_n = 1.561 \text{ s} \]
PROBLEM 19.52

A compound pendulum is defined as a rigid slab which oscillates about a fixed Point $O$ called the center of suspension. Show that the period of oscillation of a compound pendulum is equal to the period of a simple pendulum of length $OA$, where the distance from $A$ to the mass center $G$ is $GA = \bar{k}/\bar{F}$. Point $A$ is defined as the center of oscillation and coincides with the center of percussion defined in Problem 17.66.

SOLUTION

For a simple pendulum of length $OA = l$,

$$\ddot{\theta} + \frac{g}{l} \theta = 0 \quad (2)$$

Comparing Equations (1) and (2),

$$l = \frac{\bar{F}^2 + \bar{k}^2}{\bar{F}} \quad GA = l - \bar{r} = \frac{\bar{k}^2}{\bar{F}} \quad \text{Q.E.D.}$$
PROBLEM 19.53

A rigid slab oscillates about a fixed Point $O$. Show that the smallest period of oscillation occurs when the distance $r$ from Point $O$ to the mass center $G$ is equal to $k$.

SOLUTION

See Solution to Problem 19.52 for derivation of

$$\ddot{\theta} + \frac{g\bar{r}}{\bar{r}^2 + k^2} \sin \theta = 0$$

For small oscillations, $\sin \theta = \theta$ and

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{\frac{\bar{r}^2 + k^2}{g\bar{r}}}} = \frac{2\pi}{\sqrt{\bar{r} + \frac{k^2}{\bar{r}}}}$$

For smallest $\tau_n$, we must have $\bar{r} + \frac{\bar{r}^3}{\bar{r}}$ as a minimum:

$$\frac{d\left(\bar{r} + \frac{\bar{r}^3}{\bar{r}}\right)}{d\bar{r}} = 1 - \frac{k^2}{\bar{r}^2} = 0$$

$$\bar{r}^2 = k^2$$

$\bar{r} = k$ Q.E.D. \(\blacksquare\)
PROBLEM 19.54

Show that if the compound pendulum of Problem 19.52 is suspended from \( A \) instead of \( O \), the period of oscillation is the same as before and the new center of oscillation is located at \( O \).

SOLUTION

Same derivation as in Problem 19.52 with \( \bar{r} \) replaced by \( \bar{R} \).

Thus,

\[
\ddot{\theta} + \frac{g\bar{R}}{\bar{R}^2 + \bar{k}^2} \theta = 0
\]

Length of the equivalent simple pendulum is

\[
L = \frac{\bar{R}^2 + \bar{k}^2}{\bar{R}} = \bar{R} + \frac{\bar{k}^2}{\bar{R}}
\]

\[
L = (l - \bar{r}) + \frac{\bar{k}^2}{\bar{r}} = l
\]

Thus, the length of the equivalent simple pendulum is the same as in Problem 19.52. It follows that the period is the same and that the new center of oscillation is at \( O \). Q.E.D.
**PROBLEM 19.55**

The 8-kg uniform bar \( AB \) is hinged at \( C \) and is attached at \( A \) to a spring of constant \( k = 500 \, \text{N/m} \). If end \( A \) is given a small displacement and released, determine (a) the frequency of small oscillations, (b) the smallest value of the spring constant \( k \) for which oscillations will occur.

**SOLUTION**

\[
\mathcal{T} = \frac{1}{12} ml^2 = \left( \frac{1}{12} \right)(8)(0.250)^2
\]

\[\mathcal{T} = 0.04167 \, \text{kg} \cdot \text{m}^2\]

\[\alpha = \dot{\theta}\]

\[a_i = 0.04\alpha = 0.04\ddot{\theta}\]

\[\sin \theta = \theta\]

**Equation of motion.**

\[
\sum M_C = \sum (M_C)_{eff}: \quad \sqrt{-(0.165)^2 k \theta + 0.04mg \theta} = T \ddot{\theta} + (0.04)^2 m \dddot{A}
\]

\[(0.04167 + 0.01280)\dot{\theta} + (0.02722k - 0.32g)\theta = 0\]  

(1)

(a) Frequency if \( k = 500 \, \text{N} \cdot \text{m} \).

\[0.05447\ddot{\theta} + (10.47)\theta = 0\]

\[f_n = \frac{\omega_n}{2\pi} = \frac{\sqrt{10.47}}{0.05447}\]  

\[f_n = 2.21 \, \text{Hz} \Downarrow\]

(b) For \( T_n \to \infty \) or \( \omega_n \to 0 \), oscillations will not occur.

From Equation (1),

\[\omega_n^2 = \frac{0.02722k - 0.32g}{0.05447} = 0\]

\[k = \frac{0.32g}{0.02722} = (0.32)(9.81) = 30.36 \, \text{N/m} \Downarrow\]
**PROBLEM 19.56**

A 20-kg uniform square plate is suspended from a pin located at the midpoint $A$ of one of its 0.4 m edges and is attached to springs, each of constant $k = 1.6 \text{kN/m}$. If corner $B$ is given a small displacement and released, determine the frequency of the resulting vibration. Assume that each spring can act in either tension or compression.

**SOLUTION**

$$\alpha = \dot{\theta}$$

$$a_i = \frac{b}{2} \alpha - \frac{b}{2} \ddot{\theta}$$

$$\sin \theta = \theta$$

Equation of motion:

$$\zeta \Sigma M_0 = \Sigma (M_0)_{\text{eff}}: \quad - mg \frac{b}{2} \dot{\theta} + 2kb^2 \theta = T \alpha + \left( \frac{b}{2} \right)^2 m \alpha$$

$$T + m \left( \frac{b}{2} \right)^2 = \frac{1}{6} mb^2 + \frac{m b^2}{4} = \frac{5}{12} mb^2$$

$$\frac{5}{12} mb^2 \ddot{\theta} + \left( mg \frac{b}{2} + 2kb^2 \right) \theta = 0$$

$$\ddot{\theta} + \left( \frac{12g}{10b} + \frac{24k}{5m} \right) \theta = 0$$

Data:

$b = 0.4 \text{m}; \quad m = 20 \text{kg}$

$k = 1.6 \text{kN/m} = 1600 \text{N/m}$

$$\ddot{\theta} + \left[ \frac{(12)(9.81)}{(10)(0.4)} + \frac{(24)(1600)}{(5)(20)} \right] \theta = 0$$

$$\ddot{\theta} + 413.43 = 0$$

$$\omega_n^2 = 413.43 \quad \omega_n = 20.333 \text{ rad/s}$$

Frequency:

$$f_n = \frac{\omega_n}{2\pi} = \frac{20.333}{2\pi} \quad f_n = 3.24 \text{ Hz} \swarrow$$
PROBLEM 19.57

Two uniform rods, each of mass $m = 12$ kg and length $L = 800$ mm, are welded together to form the assembly shown. Knowing that the constant of each spring is $k = 500$ N/m and that end $A$ is given a small displacement and released, determine the frequency of the resulting motion.

SOLUTION

Equation of motion. \[\Sigma M_0 = \Sigma (M_0)_{\text{eff}}: \quad + \left[ m_{AC} g \frac{L}{2} - 2k \left( \frac{L}{2} \right)^2 \right] \theta = (T_{AC} + T_{BD}) \alpha + m_{AC} \left( \frac{L}{2} \right)^2 \alpha \]

\[m_{BD} = m_{AC} = m\]

\[T_{BD} = T_{BD} = T = \frac{1}{2} mL^2\]

\[ \left( \frac{1}{6} + \frac{1}{4} \right) m L^2 \ddot{\theta} + \left[ 2k \left( \frac{L}{2} \right)^2 - mg \frac{L}{2} \right] \theta = 0\]

Data:

\[L = 800 \text{ mm} = 0.8 \text{ m}, \quad m = 12 \text{ kg}, \quad k = 500 \text{ N/m}\]

Frequency:

\[\omega_n^2 = \frac{6((500)(0.8) - (12)(9.81))}{(5)(12)(0.8)} = 35.285\]

\[\omega_n = 5.9401 \text{ rad/s} \quad f_n = \frac{\omega_n}{2\pi} \quad f_n = 0.945 \text{ Hz} \]

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PROBLEM 19.58

The rod \( ABC \) of total mass \( m \) is bent as shown and is supported in a vertical plane by a pin at \( B \) and by a spring of constant \( k \) at \( C \). If end \( C \) is given a small displacement and released, determine the frequency of the resulting motion in terms of \( m \), \( L \), and \( k \).

SOLUTION

\[ \alpha = \ddot{\theta} \quad \ddot{a}_i = \frac{L}{2} \dddot{\theta} \]

\[ \bar{T} = \frac{1}{12} \frac{mL^2}{2} \]

\[ \sin \theta = \theta \quad \cos \theta = 1 \]

Equation of motion.

\[
\gamma \sum M_B = \Sigma (H_B)_{eq} : \quad \frac{mg}{2} \frac{L}{2} \sin \theta + \frac{mg}{2} \frac{L}{2} \cos \theta - kL(L\theta + \delta_{ST}) \cos \theta = 2\bar{T} \ddot{\theta} + 2 \frac{m}{2} \frac{\dddot{a}_i}{2} \frac{L}{2} - kL^2 \delta_{ST} = \frac{mL^2}{12} \dddot{\theta} + \frac{mL^2}{4} \dddot{\theta} \]

But for equilibrium (\( \theta = 0 \)),

\[
\sum M_B = 0 = \frac{m}{2} \frac{gL}{2} - kL^2 \delta_{ST} \]
PROBLEM 19.58 (Continued)

Substituting Eq. (2) into Eq. (1),

\[
\left(\frac{mgL}{4} - kl^2\right)\ddot{\theta} = \frac{ml^2}{3} \ddot{\theta} \\
\ddot{\theta} + \left(\frac{kl^2 - \frac{mgL}{4}}{\frac{ml^2}{3}}\right)\ddot{\theta} = 0
\]

Frequency,

\[
\omega_n^2 = \frac{3k}{m} - \frac{3g}{4L} \\
\omega_n = \sqrt{\frac{k}{m} - \frac{g}{4L}} \\
f_n = \frac{\omega_n}{2\pi} \\
f_n = \frac{\sqrt{\frac{k}{m} - \frac{g}{4L}}}{2\pi}
\]
PROBLEM 19.59

A uniform disk of radius \( r = 250 \text{ mm} \) is attached at \( A \) to a 650-mm rod \( AB \) of negligible mass which can rotate freely in a vertical plane about \( B \). If the rod is displaced \( 2^\circ \) from the position shown and released, determine the magnitude of the maximum velocity of Point \( A \), assuming that the disk \( (a) \) is free to rotate in a bearing at \( A \), \( (b) \) is riveted to the rod at \( A \).

SOLUTION

\[ T = \frac{1}{2} mr^2 \]

Kinematics:

\[ \alpha = \dot{\theta} \]

\[ a_t = l \alpha = l \dot{\theta} \]

\((a)\) The disk is free to rotate and is in curvilinear translation.

Thus,

Equation of motion.

\[ \sum M_B = (\sum M_B)_{\text{eff}}: \quad -mg l \sin \theta = I \alpha_t, \quad \sin \theta = \theta \]

\[ ml^2 \ddot{\theta} + mgl \theta = 0 \]

Frequency:

\[ \omega_n^2 = \frac{g}{l} \]

\[ = \frac{9.81}{0.650} \]

\[ = 15.092 \]

\[ \omega_n = 3.8849 \text{ rad/s} \]

\[ \tau_n = \frac{2\pi}{\omega_n} = 1.617 \text{ s} \]

\( \tau_n = 1.617 \text{ s} \)

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PROBLEM 19.59 (Continued)

(b) When the disk is riveted at \( A \), it rotates at an angular acceleration \( \alpha \).

Equation of motion: \( \Sigma M_B = (\Sigma M_B)_{\text{eff}}: \quad -mg\ell \sin \theta = T\alpha + Ima, \quad T = \frac{1}{2}mr^2, \quad \sin \theta = \theta \)

\[
\left( \frac{1}{2}mr^2 + ml^2 \right) \ddot{\theta} + mg\ell \theta = 0
\]

Frequency:

\[
\omega_n^2 = \frac{gl}{\left( \frac{r^2}{2} + l^2 \right)} = \frac{(9.81)(0.650)}{\frac{1}{2}(0.250)^2 + (0.650)^2} = 14.053
\]

\( \omega_n = 3.7487 \text{ rad/s} \)

\[
\tau_n = \frac{2\pi}{\omega_n} = 1.676 \text{ s} \quad \tau_n = 1.676 \text{ s}
\]
PROBLEM 19.60

A 3-kg slender rod is suspended from a steel wire which is known to have a torsional spring constant \(K = 2.25 \text{ N} \cdot \text{m/rad}\). If the rod is rotated through 180° about the vertical and released, determine (a) the period of oscillation, (b) the maximum velocity of end \(A\) of the rod.

SOLUTION

Equation of motion:

\[
\Sigma M_G = \Sigma (M_G)_{eq}: \quad -K\theta = T\ddot{\theta} + \frac{K}{I} \theta = 0
\]

\[
\ddot{\theta} + \omega_n^2 \theta = 0 \quad \omega_n^2 = \frac{K}{I}
\]

Data:

\[
m = 3 \text{ kg} \\
l = 200 \text{ mm} = 0.2 \text{ m} \\
I = \frac{1}{12} ml^2 = \frac{1}{12} (3)(0.2)^2 \\
= 0.01 \text{ kg} \cdot \text{m}^2 \\
K = 2.25 \text{ N} \cdot \text{m/rad} \\
\omega_n^2 = 2.25 \\
\omega_n = 15 \text{ rad/s}
\]

(a) Period of oscillation.

\[
\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{15}
\]

\[
\tau_n = 0.419 \text{ s} \quad \therefore
\]

Simple harmonic motion:

\[
\theta = \theta_m \sin(\omega_n t + \phi) \\
\dot{\theta} = \omega_n \theta_m \cos(\omega_n t + \phi) \\
\ddot{\theta} = \omega_n \theta_m
\]

\[
(v_A)_m = \frac{l}{2} \dot{\theta}_m = \frac{1}{2} \omega_n \theta_m \\
\theta_m = 180^\circ = \pi \text{ radians}
\]

(b) Maximum velocity at end \(A\).

\[
(v_A)_m = \left(\frac{1}{2}\right)(0.2)(15)(\pi) \\
(v_A)_m = 4.71 \text{ m/s} \quad \therefore
\]
**PROBLEM 19.61**

A homogeneous wire bent to form the figure shown is attached to a pin support at A. Knowing that \( r = 220 \text{ mm} \) and that Point \( B \) is pushed down 20 mm and released, determine the magnitude of the velocity of \( B \), 8 s later.

**SOLUTION**

Determine location of the centroid \( G \).

Let \( \rho = \text{mass per unit length} \)

Then total mass

\[
m = \rho(2r + \pi r) = \rho(2 + \pi) r
\]

About \( C \):

\[
m g c = 0 + \pi r \rho \left( \frac{2r}{\pi} \right) g = 2r^2 \rho g
\]

\[
\bar{y} = \frac{2r}{\pi} \quad \text{for a semicircle}
\]

\[
\rho r(2 + \pi) c = 2r^2 \rho, \quad c = \frac{2r}{(2 + \pi)}
\]

**Equation of motion.**

\[
\sum M_0 = \Sigma(M_0)_{eff}: \quad \alpha = \dot{\theta} \quad \alpha_c = c \alpha = c \dot{\theta}
\]

\[
-m g c \sin \theta = I \ddot{\alpha} + m c a\sin \theta
\]

\[
(T + m c^2) \ddot{\theta} + m g c \theta = 0 \quad I_0 \dot{\theta} + m g c \theta = 0
\]

**Moment of inertia.**

\[
\bar{I} + m c^2 = I_0
\]

\[
I_0 = (I_0)_{\text{semicirc.}} + (I_0)_{\text{line}} = m_{\text{semicirc.}} r^2 + m_{\text{line}} \left( \frac{(2r)^2}{12} \right)
\]

\[
m_{\text{semicirc.}} = \rho \pi r \quad m_{\text{line}} = 2 \rho r \quad \rho = \frac{m}{(2 + \pi)r}
\]

\[
I_0 = \rho \left[ \pi r^2 \cdot r + 2r^2 \frac{r^2}{3} \right] = \frac{m r^2}{(2 + \pi)} \left[ \frac{r}{3} + \frac{2}{3} \right]
\]

\[
\frac{m r^2}{(2 + \pi)} \left[ \frac{r}{3} + \frac{2}{3} \right] \dot{\theta} + m g \frac{2r}{(2 + \pi)} \theta = 0
\]
PROBLEM 19.61 (Continued)

**Frequency.**

\[ \omega_n = \frac{2g}{(\pi + \frac{2}{3})r} = \frac{(2)(9.81)}{(\pi + \frac{2}{3})(0.220)} \]

\[ \omega_n = 23.418 \text{ s}^{-2} \quad \omega_n = 4.8392 \text{ rad/s} \]

\[ \theta = \theta_m \sin(\omega_n t + \phi) \quad y_B = r\theta \]

At \( t = 0, \)

\[ y_B = 20 \text{ mm}, \quad \dot{y}_B = 0 \]

\[ \ddot{y}_B = (y_B)_m \omega_n \cos(0 + \phi), \quad \phi = \frac{\pi}{2} \]

\[ y_B = 20 \text{ mm} = (y_B)_m \sin\left(0 + \frac{\pi}{2}\right), \quad (y_B)_m = 20 \text{ mm} \]

\[ \ddot{y}_B = (20 \text{ mm})\sin\left(\omega_n t + \frac{\pi}{2}\right) \quad \omega_n = 4.8392 \text{ rad/s} \]

\[ \dddot{y}_B = 20\omega_n \cos\left(\omega_n t + \frac{\pi}{2}\right) = -(20 \text{ mm})\omega_n \sin\omega_n t \]

At \( t = 8 \text{ s}, \)

\[ \dddot{y}_B = -(20)(4.8392)\sin[(4.8392)(8)] = -(96.88)(0.8492) \]

\[ = -82.2 \text{ mm/s} \quad v_B = 82.2 \text{ mm/s} \uparrow \]
**PROBLEM 19.62**

A homogeneous wire bent to form the figure shown is attached to a pin support at \( A \). Knowing that \( r = 400 \text{ mm} \) and that Point \( B \) is pushed down 40 mm and released, determine the magnitude of the acceleration of \( B \), 10 s later.

**SOLUTION**

Determine location of the centroid \( G \).

Let \( \rho = \text{mass per unit length} \)

Then total mass \( m = \rho(2r + \pi r) = \rho r(2 + \pi) \)

About \( C \):

\[
mgc = 0 + \pi r \rho \left( \frac{2r}{r} \right) g = 2r^2 \rho g
\]

\[
\bar{y} = \frac{2r}{2} \quad \text{for a semicircle}
\]

\[
\rho r(2 + \pi)c = 2r^2 \rho \quad c = \frac{2r}{(2 + \pi)}
\]

Equation of motion.

\[
\sum M_0 = \Sigma (M_0)_{\text{eff}}: \quad \alpha = \dot{\theta} \quad a_i = c \alpha = c \dot{\theta}
\]

\[
-mgc \sin \theta = I \alpha + mca_n \quad \sin \theta = \theta
\]

\[
(T + mc^2) \ddot{\theta} + mgc \theta = 0 \quad I_0 \ddot{\theta} + mgc \theta = 0
\]

Moment of inertia.

\[
I_0 + mc^2 = I_0
\]

\[
I_0 = (I_0)_{\text{semicirc.}} + (I_0)_{\text{line}} = m_{\text{semicirc.}} r^2 + m_{\text{line}} \left( \frac{2r^2}{12} \right)
\]

\[
m_{\text{semicirc.}} = \rho \pi r \quad m_{\text{line}} = 2 \rho r \quad \rho = \frac{m}{r(2 + \pi)}
\]

\[
I_0 = \rho \left[ \pi r \cdot r^2 + \frac{2r \cdot r^2}{3} \right] = \frac{mr^2}{(2 + \pi)} \left[ \pi + \frac{2}{3} \right]
\]

\[
\frac{mr^2}{(2 + \pi)} \left[ \pi + \frac{2}{3} \right] \ddot{\theta} + mg \frac{2r}{(2 + \pi)} \theta = 0
\]
PROBLEM 19.62 (Continued)

Frequency:

\[
\omega_n^2 = \frac{2g}{(\pi + \frac{2}{3})r} = \frac{(2)(9.81)}{(\pi + \frac{2}{3})(0.4)}
\]

\[
\omega_n^2 = 12.8799 \text{ (rad/s)}^2 \quad \omega_n = 3.5889 \text{ rad/s}
\]

\[
\theta = \theta_m \sin(\omega_n t + \phi) \quad y_B = r\theta
\]

\[
y_B = r\theta_m \sin(\omega_n t + \phi) = (y_B)_m \sin(\omega_n t + \phi)
\]

At \( t = 0, \)

\[
y_B = 40 \text{ mm} \quad \dot{y}_B = 0
\]

\((t = 0): \quad \dot{y}_B = 0 = (y_B)_m \cos(0 + \phi) \quad \phi = \frac{\pi}{2}
\]

\[
y_B = 40 \text{ mm} = (y_B)_m \sin \left(0 + \frac{\pi}{2}\right) \quad (y_B)_m = 40 \text{ mm}
\]

\[
y_B = (40 \text{ mm}) \sin \left(\omega_n t + \frac{\pi}{2}\right) \quad \omega_n = 3.5889 \text{ rad/s}
\]

\[
\dot{y}_B = (40)(\omega_n) \cos \left(\omega_n t + \frac{\pi}{2}\right) = -(40)\omega_n \sin \omega_n t
\]

\[
\ddot{y}_B = (-40)\left(\omega_n^2\right) \sin \left(\omega_n t + \frac{\pi}{2}\right) = 40\omega_n^2 \cos \omega_n t
\]

At \( t = 10 \text{ s}, \)

\[
(a_B)_B = \ddot{y}_B = (-40)(3.5889)^2 \cos(3.5889)(10) = -122.124 \text{ mm/s}^2
\]

\[
(v_B)_B = \dot{y}_B = -(40)(3.5889) \sin \left[(3.5889)(10)\right] = -139.465 \text{ mm/s}
\]

\[
a_B = \left[\left(\frac{a_B}{r}\right)^2 + \left(\frac{v_B^2}{r}\right)^2\right]^{1/2} = \left[(122.124)^2 + \left(\frac{(-139.465)^2}{400}\right)^2\right]^{1/2} = 122.3 \text{ mm/s}^2 \text{ } \text{ ▽}
\]
PROBLEM 19.63

A uniform disk of radius \( r = 120 \text{ mm} \) is welded at its center to two elastic rods of equal length with fixed ends at \( A \) and \( B \). Knowing that the disk rotates through an \( 8^\circ \) angle when a 500-mN·m couple is applied to the disk and that it oscillates with a period of 1.3 s when the couple is removed, determine (\( a \)) the mass of the disk, (\( b \)) the period of vibration if one of the rods is removed.

SOLUTION

Torsional spring constant.

\[
k = \frac{T}{\theta} = \frac{0.5 \text{ N} \cdot \text{m}}{(8)(\frac{\pi}{180})} = 3.581 \text{ N} \cdot \text{m/} \text{rad}
\]

Equation of motion.

\[
\Sigma M_0 = \Sigma (M_0)_{\text{eff}} : -K\theta = J\ddot{\theta} + \frac{K}{J}\theta = 0
\]

Natural frequency and period.

\[
\omega_n^2 = \frac{K}{J}
\]

Period.

\[
\tau_n = \frac{2\pi}{\omega_n} = 2\pi \sqrt{\frac{J}{K}}
\]

Mass moment of inertia.

\[
J = \frac{\tau_n^2K}{(2\pi)^2} = \frac{(1.35)^2(3581 \text{ N} \cdot \text{m/r})}{(2\pi)^2}
\]

\[
J = 0.1533 \text{ N} \cdot \text{m} \cdot \text{s}^2 = \frac{1}{2}mr^2 = \frac{1}{2}m(0.120 \text{ m})^2
\]

(\( a \)) Mass of the disk.

\[
m = \frac{(0.1533 \text{ N} \cdot \text{m} \cdot \text{s}^2)(2)}{(0.120 \text{ m})^2}
\]

\[m = 21.3 \text{ kg} \quad \square\]

(\( b \)) With one rod removed:

\[
K' = \frac{K}{2} = \frac{3.581}{2} = 1.791 \text{ N} \cdot \text{m/} \text{rad}
\]

Period.

\[
\tau = 2\pi \sqrt{\frac{J}{K'}} = 2\pi \sqrt{\frac{(0.1533 \text{ N} \cdot \text{m} \cdot \text{s}^2)}{1.791 \text{ N} \cdot \text{m/} \text{rad}}}
\]

\[\tau = 1.838 \text{ s} \quad \square\]
PROBLEM 19.64

A 5 kg uniform rod $CD$ of length $l = 0.75$ m is welded at $C$ to two elastic rods, which have fixed ends at $A$ and $B$ and are known to have a combined torsional spring constant $K = 30 \text{ N} \cdot \text{m/rad}$. Determine the period of small oscillation if the equilibrium position of $CD$ is (a) vertical as shown, (b) horizontal.

SOLUTION

(a) Equation of motion.

$$\alpha = \ddot{\theta} \quad a_r = \frac{l}{2}\alpha = \frac{l}{2}\ddot{\theta}$$

$$\sum M_C = (M_C)_{\text{eff}}: \quad -K\theta - \frac{1}{2}mg\frac{l}{2}\sin\theta = I\alpha + \frac{1}{2}(ma_t)$$

$$-K\theta - \frac{1}{2}mg\frac{l}{2}\sin\theta = \ddot{\theta} + \frac{1}{4}ml^2\dddot{\theta}$$

$$\left(\dddot{\theta} + \frac{1}{4}ml^2\right)\dddot{\theta} + \frac{1}{2}mg\frac{l}{2}\sin\theta + K\theta = 0$$

$$\left(\frac{1}{12}ml^2 + \frac{1}{4}ml^2\right)\dddot{\theta} + K\theta + \frac{1}{2}mg\frac{l}{2}\theta = 0$$

$$\frac{1}{3}ml^2\dddot{\theta} + \left(K + \frac{1}{2}mg\frac{l}{2}\right)\theta = 0$$

$$\dddot{\theta} + \left(\frac{3K}{ml^2} + \frac{3g}{2l}\right)\theta = 0$$
PROBLEM 19.64 (Continued)

Data:

\[ K = 30 \text{ N} \cdot \text{m/rad}, \quad m = 5 \text{ kg}, \quad l = 0.75 \text{ m} \]

\[ \ddot{\theta} + \left[ \frac{(3)(30)}{(5)(0.75)^2} + \frac{(3)(9.81)}{(2)(0.75)} \right] \theta = 0 \]

\[ \dot{\theta} + 51.62 \theta = 0 \]

Frequency.

\[ \omega_n^2 = 51.62 \quad \omega_n = 7.1847 \text{ rad/s} \]

Period.

\[ \tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{7.1847} \quad \tau_n = 0.875 \text{ s} \]

(b) If the rod is horizontal, the gravity term is not present and the equation of motion is

\[ \ddot{\theta} + \frac{3K}{ml^2} \theta = 0 \]

\[ \omega_n^2 = \frac{3K}{ml^2} = \frac{(3)(30)}{(5)(0.75)^2} = 32 \]

\[ \omega_n = 5.9938 \text{ rad/s} \quad \tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{5.65685} \quad \tau_n = 1.111 \text{ s} \]
PROBLEM 19.65

A 1.8-kg uniform plate in the shape of an equilateral triangle is suspended at its center of gravity from a steel wire which is known to have a torsional constant \( K = 35 \text{ mN} \cdot \text{m/rad} \). If the plate is rotated 360° about the vertical and then released, determine (a) the period of oscillation, (b) the maximum velocity of one of the vertices of the triangle.

SOLUTION

Mass moment of inertia of plate about a vertical axis:

\[
h = \frac{\sqrt{3}}{2} b, \quad A = \frac{1}{2} bh = \frac{\sqrt{3}}{4} b^2
\]

For area,

\[
T_x = T_y = \frac{1}{36} bh^3 = \frac{\sqrt{3} b^4}{96}
\]

\[
T_z = T_x + T_y = \frac{\sqrt{3}}{48} b^4
\]

For mass,

\[
T = \frac{m}{A} (T_z)_{\text{area}} = \left( \frac{4 m}{\sqrt{3} b} \right) \left( \frac{\sqrt{3}}{48} b^4 \right) = \frac{1}{12} mb^2
\]

\[
T = \frac{1}{12} (1.8)(0.150)^2 = 3.375 \times 10^{-3} \text{ kg} \cdot \text{m}^2
\]

Equation of motion:

\[
\sum M_G = \sum (M_G)_{\text{eff}} : \quad -K \theta = T \theta
\]

\[
\ddot{\theta} + \frac{K}{T} \theta = 0
\]

Frequency:

\[
\omega_n^2 = \frac{K}{T} = \frac{35 \times 10^{-3}}{3.375 \times 10^{-3}} = 10.37
\]

\[
\omega_n = 3.2203 \text{ rad/s}
\]
PROBLEM 19.65 (Continued)

(a) Period.
\[
\tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{3.2203} \quad \tau = 1.951 \text{ s}
\]

Maximum rotation.
\[
\theta_m = 360^\circ = 2\pi \text{ rad}
\]

Maximum angular velocity.
\[
\dot{\theta}_m = \omega_n \theta_m = (3.2203)(2\pi) = 20.234 \text{ rad/s}
\]

(b) Maximum velocity at a vertex.
\[
v_m = r\dot{\theta}_m = \frac{2}{3}h\dot{\theta}_m = \frac{2}{3} \frac{\sqrt{3}}{2} b = \left(\frac{2}{3}\right)\left(\frac{\sqrt{3}}{2}\right)(0.150)(20.234)
\]
\[
v_m = 1.752 \text{ m/s}
\]
PROBLEM 19.66

A horizontal platform $P$ is held by several rigid bars which are connected to a vertical wire. The period of oscillation of the platform is found to be 2.2 s when the platform is empty and 3.8 s when an object $A$ of unknown moment of inertia is placed on the platform with its mass center directly above the center of the plate. Knowing that the wire has a torsional constant $K = 30 \, \text{N} \cdot \text{m/rad}$, determine the centroidal moment of inertia of object $A$.

SOLUTION

Equation of motion:  
$$
\sum M_G = \sum (M_G)_{\text{eff}}: \quad -K\theta = \ddot{T}
$$

$$
\dot{\theta} + \frac{K}{I} \theta = 0 \quad \omega_n^2 = \frac{K}{I}
$$

Case 1. The platform is empty.

$$
\omega_{n1} = \frac{2\pi}{T_1} = \frac{2\pi}{2.2} = 2.856 \, \text{rad/s}
$$

$$
T_1 = \frac{K}{\omega_{n1}^2} = \frac{30}{(2.856)^2} = 3.6779 \, \text{kg} \cdot \text{m}^2
$$

Case 2. Object $A$ is on the platform.

$$
\omega_{n2} = \frac{2\pi}{T_2} = \frac{2\pi}{3.8} = 1.653 \, \text{rad/s}
$$

$$
T_2 = \frac{K}{\omega_{n2}^2} = \frac{30}{(1.653)^2} = 10.9793 \, \text{kg} \cdot \text{m}^2
$$

Moment of inertia of object $A$.

$$
\bar{T}_A = T_2 - T_1 \quad \bar{T}_A = 7.30 \, \text{kg} \cdot \text{m}^2
$$
PROBLEM 19.67

A thin rectangular plate of sides \( a \) and \( b \) is suspended from four vertical wires of the same length \( l \). Determine the period of small oscillations of the plate when (a) it is rotated through a small angle about a vertical axis through its mass center \( G \), (b) it is given a small horizontal displacement in a direction perpendicular to \( AB \), (c) it is given a small horizontal displacement in a direction perpendicular to \( BC \).

SOLUTION

(a) Plate is rotated about vertical axis.

Similarly, for \( B, D, \) and \( C \),

\[
\Sigma M_G = \Sigma (M_G)_{eff} : \quad -4T \frac{r}{l} \cdot r = J \ddot{\theta} \quad T = \frac{mg}{4}
\]

\[
\ddot{\theta} = \frac{mgr^2}{Jl} \theta = 0 \quad J = \frac{1}{12} m (a^2 + b^2)
\]

\[
r^2 = \frac{1}{4} (a^2 + b^2)
\]

\[
\tau_n = \frac{2\pi}{\omega_n} = 2\pi \sqrt{\frac{Jl}{mgr^2}}
\]

\[
\tau_n = 2\pi \sqrt{\frac{1}{12} m (a^2 + b^2) l}{mg \frac{1}{4} (a^2 + b^2)} = 2\pi \sqrt{\frac{l}{3g}}
\]

(b) Plate is moved horizontally. The plate is in curvilinear translation.

\[
\sin \theta = \theta \quad \cos \theta = 1 \quad l \ddot{\theta} \cos \theta = l \ddot{\theta} = a
\]
PROBLEM 19.67 (Continued)

\[ \sum F_y = 0 = 4(T \cos \theta) - mg = 0 \]
\[ T = \frac{mg}{4} \]
\[ + \sum F_H = \Sigma(F_H)_\text{eff} : -4T \sin \theta = m\ddot{\theta} \]
\[ m\ddot{\theta} + g\theta = 0 \]
\[ \tau_n = 2\pi \sqrt{\frac{I}{mg}} \]

(c) Since the oscillation about axes parallel to AB (and CD) is independent of the length of the sides of the plate, the period of vibration about axes parallel to BC (and AD) is the same.

\[ \tau_n = 2\pi \sqrt{\frac{I}{mg}} \]
PROBLEM 19.68

A circular disk of radius \( r = 0.8 \text{ m} \) is suspended at its center \( C \) from wires \( AB \) and \( BC \) soldered together at \( B \). The torsional spring constants of the wires are \( K_1 = 100 \text{ N} \cdot \text{m/rad} \) for \( AB \) and \( K_2 = 50 \text{ N} \cdot \text{m/rad} \) for \( BC \). If the period of oscillation is 0.5 s about the axis \( AC \), determine the mass of the disk.

SOLUTION

Spring constant. Let \( T \) be the torque carried by the wires.

\[
\theta_{B/A} = \frac{T}{K_1} \quad \theta_{C/B} = \frac{T}{K_2} \\
\theta_{C/A} = \theta_{B/A} + \theta_{C/B} = \left( \frac{1}{K_1} + \frac{1}{K_2} \right) T = \frac{T}{K_{eq}} \\
\frac{1}{K_{eq}} = \frac{1}{K_1} + \frac{1}{K_2} \quad K_{eq} = \frac{K_1 K_2}{K_1 + K_2} = \frac{(100)(50)}{100 + 50} = 33.333 \text{ N} \cdot \text{m/rad}
\]

Equation of motion.

\[
\Sigma M_C = \Sigma (M_C)_{eq}: -K_{eq} \theta = \ddot{\theta} \\
\ddot{\theta} + \frac{K_{eq}}{I} \theta = 0 \\
\omega_n^2 = \frac{K_{eq}}{I} \quad \text{or} \quad \ddot{T} = \frac{K_{eq}}{\omega_n^2}
\]

But \( \tau_n = 0.5 \text{ s} \):

\[
\omega_n = \frac{2\pi}{\tau_n} = \frac{2\pi}{0.5} = 12.566 \text{ rad/s}
\]

Then

\[
\ddot{T} = \frac{33.333}{(12.566)^2} = 0.21109 \text{ kg} \cdot \text{m}^2
\]

For a circular disk,

\[
\ddot{T} = \frac{1}{2} mr^2 \\
m = \frac{2\ddot{T}}{r^2} = \frac{(2)(0.21109)}{(0.8)^2} \quad m = 0.660 \text{ kg}
\]
PROBLEM 19.69

Determine the period of small oscillations of a small particle which moves without friction inside a cylindrical surface of radius $R$.

SOLUTION

Datum at ②:

Position ①

$$T_1 = 0$$
$$V_1 = WR(1 - \cos \theta_m)$$

Small oscillations:

$$(1 - \cos \theta_m) = 2 \sin^2 \frac{\theta_m}{2} = \frac{\theta_m^2}{2}$$

$$V_1 = \frac{WR\theta_m^2}{2}$$

Position ②

$$v_m = R \dot{\theta}_m$$
$$T_2 = \frac{1}{2} mv_m^2 = \frac{1}{2} mR^2 \dot{\theta}_m^2$$

$$V_2 = 0$$

Conservation of energy:

$$T_1 + V_1 = T_2 + V_2$$
$$0 + WR \frac{\theta_m^2}{2} = \frac{1}{2} mR^2 \dot{\theta}_m^2 + 0$$

$$\dot{\theta}_m = \omega_n \theta_m$$

$$W = mg$$

$$mgR \frac{\theta_m^2}{2} = \frac{1}{2} mR^2 \omega_n^2 \theta_m^2$$

$$W_n = \sqrt{\frac{g}{R}}$$

Period of oscillations:

$$\tau_n = \frac{2\pi}{\omega_n} = 2\pi \sqrt{\frac{R}{g}}$$
**PROBLEM 19.70**

A 400 g sphere \( A \) and a 300 g sphere \( C \) are attached to the ends of a rod \( AC \) of negligible weight which can rotate in a vertical plane about an axis at \( B \). Determine the period of small oscillations of the rod.

**SOLUTION**

Datum at \( \textcircled{1} \):

**Position \( \textcircled{1} \)**

\[
T_1 = 0 \\
V_1 = W_C h_C - W_A h_A \\
h_C = BC(1 - \cos \theta_m) \\
h_A = BA(1 - \cos \theta_m)
\]

Small angles.

\[
1 - \cos \theta_m = \frac{\theta_m^2}{2}
\]

\[
V_1 = [(m_C g)(BC) - (m_A g)(BA)] \frac{\theta_m^2}{2}
\]

\[
V_1 = [(0.3 \text{ kg})(0.2 \text{ m}) - (0.4 \text{ kg})(0.125 \text{ m})] \left( \frac{\theta_m^2}{2} \right) (9.81 \text{ m/s}^2)
\]

\[
V_1 = (0.0981)
\]

**Position \( \textcircled{2} \)**

\[
T_2 = \frac{1}{2} m_C (v_0)^2_m + \frac{1}{2} m_A (v_A)^2_m, \quad (v_C)_m = 0.2 \dot{\theta}_m
\]

\[
T_2 = \frac{1}{2} m_C (0.2)^2 \dot{\theta}_m^2 + \frac{1}{2} m_A (0.125)^2 \dot{\theta}_m^2, \quad (v_A)_m = 0.125 \dot{\theta}_m
\]

\[
T_2 = \frac{1}{2} [(0.3)(0.2)^2 + (0.4)(0.125)^2] W_n^2 \theta_m^2, \quad \dot{\theta}_m^2 = \omega_n \theta_m
\]

\[
T_2 = \frac{1}{2} (0.01825) \omega_n^2 \theta_m^2
\]
PROBLEM 19.70 (Continued)

Conservation of energy: \( T_1 + V_1 = T_2 + V_2 : \quad 0 + 0.0981 \left( \frac{\theta_m^2}{2} \right) = 0.01825 \omega_n^2 \theta_m^2 \)

\[ \omega_n^2 = \frac{(0.0981)}{(0.01825)} = 5.3753 \]

Period of oscillations: \[ \tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{5.3753}} \]

\[ \tau_n = 2.71 \text{ s} \]
PROBLEM 19.71

A 1.8-kg collar $A$ is attached to a spring of constant 800 N/m and can slide without friction on a horizontal rod. If the collar is moved 70 mm to the left from its equilibrium position and released, determine the maximum velocity and maximum acceleration of the collar during the resulting motion.

SOLUTION

Datum at $\odot$:

Position $\odot$

\[ T_1 = 0 \quad V_1 = \frac{1}{2} kx_m^2 \]

\[ \dot{x} = \omega_n x_m \]

Position $\odot$

\[ T_2 = \frac{1}{2} mv_2^2 \quad V_2 = 0 \quad v_2 = \dot{x}_m \]

\[ T_1 + V_1 = T_2 + V_2 \quad 0 + \frac{1}{2} kx_m^2 = \frac{1}{2} mx_m^2 + 0 \]

\[ \frac{1}{2} kx_m^2 = \frac{1}{2} m\omega_n^2 x_m^2 \quad \omega_n^2 = \frac{k}{m} = \frac{800 \text{ N/m}}{1.8 \text{ kg}} \]

\[ \omega_n^2 = 444.4 \text{ s}^{-2} \quad \omega_n = 21.08 \text{ rad/s} \]

\[ \dot{x}_m = \omega_n x_m = (21.08 \text{ s}^{-1})(0.070 \text{ m}) = 1.476 \text{ m/s} \]

\[ \ddot{x}_m = \omega_n^2 x_m = (21.08 \text{ s}^{-2})(0.070 \text{ m}) = 1.51 \text{ m/s}^2 \]
PROBLEM 19.72

A 1.5-kg collar $A$ is attached to a spring of constant 1 kN/m and can slide without friction on a horizontal rod. The collar is at rest when it is struck with a mallet and given an initial velocity of 1 m/s. Determine the amplitude of the resulting motion and the maximum acceleration of the collar.

SOLUTION

$m = 1.5 \text{ kg} \quad k = 1 \text{kN/m} = 1000 \text{ N/m}$

Position 1: Immediately after collar is struck:

$x = 0, \quad v = 1 \text{ m/s}$

$V_1 = 0, \quad T_1 = \frac{1}{2} mv^2 = \frac{1}{2} (1.5)(1)^2 = 0.75 \text{ N} \cdot \text{m}$

Position 2: $v = 0, x$ is maximum, i.e., $x = x_m$

$V_2 = \frac{1}{2} kx_m^2 = \frac{1}{2} (1000)x_m^2 = 500x_m^2 \quad T_2 = 0$

Conservation of energy.

$T_1 + V_1 = T_2 + V_2$

$0.75 + 0 = 0 + 500x_m^2$

Amplitude of motion.

$x_m = 0.03873 \text{ m}$

$x_m = 38.7 \text{ mm}$

Maximum force:

$F_m = kx_m = (1000)(0.03873) = 38.73 \text{ N}$

Maximum acceleration.

$a_m = \frac{F_m}{m} = \frac{38.73}{1.5} = 25.82 \text{ m/s}^2$

$a_m = 25.8 \text{ m/s}^2$

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PROBLEM 19.73

A uniform rod $AB$ can rotate in a vertical plane about a horizontal axis at $C$ located at a distance $c$ above the mass center $G$ of the rod. For small oscillations, determine the value of $c$ for which the frequency of the motion will be maximum.

SOLUTION

Find $\omega_n$ as a function of $c$.

Datum at $\odot$:

Position $\odot$

$T_1 = 0 \quad V_1 = mgh$

$V_1 = mgc(1 - \cos \theta_m)$

$1 - \cos \theta_m = 2 \sin^2 \frac{\theta_m}{2} = \frac{\theta_m^2}{2}$

$V_1 = mgc \frac{\theta_m^2}{2}$

$T_2 = \frac{1}{2} I_c \theta_m^2$

$I_C = I + mc^2 = \frac{1}{12} ml^2 + mc^2$

$T_2 = \frac{1}{2} m \left( \frac{l^2}{12} + c^2 \right) \theta_m^2 \quad V_2 = 0$

$T_1 + V_1 = T_2 + V_2 \quad 0 + mgc \frac{\theta_m^2}{2} = m \left( \frac{l^2}{12} + c^2 \right) \frac{\theta_m^2}{2} + 0$

$\dot{\theta}_m = \omega_n \theta_m$

$gc = m \left( \frac{l^2}{12} + c^2 \right) \omega_n^2$

$\omega_n^2 = \frac{gc}{\left( \frac{l^2}{12} + c^2 \right)}$

Maximum $c$, when

$\frac{d\omega_n^2}{dc} = 0 = \frac{g \left( \frac{l^2}{12} + c^2 \right) - 2c^2 g}{\left( \frac{l^2}{12} + c^2 \right)} = 0$

$\frac{l^2}{12} - c^2 = 0$

$c = \frac{l}{\sqrt{12}}$
PROBLEM 19.74

A homogeneous wire of length \(2l\) is bent as shown and allowed to oscillate about a frictionless pin at \(B\). Denoting by \(\tau_0\) the period of small oscillations when \(\beta = 0\), determine the angle \(\beta\) for which the period of small oscillations is \(2\tau_0\).

SOLUTION

We denote by \(m\) the mass of half the wire.

\[
\text{Position 1} \quad \text{Maximum deflections:}
\]

\[T_1 = 0, \quad V_1 = -mg \frac{l}{2} \cos(\theta_m - \beta) - mg \frac{l}{2} \cos(\theta_m + \beta)\]

\[= -mg \frac{l}{2} (\cos \theta_m \cos \beta + \sin \theta_m \sin \beta + \cos \theta_m \cos \beta - \sin \theta_m \sin \beta)\]

\[V_1 = -mgl \cos \beta \cos \theta_m\]

For small oscillations, \(\cos \theta_m = 1 - \frac{1}{2} \theta_m^2\)

\[V_1 = -mgl \cos \beta + \frac{1}{2} mgl \cos \beta \theta_m^2\]

\[
\text{Position 2} \quad \text{Maximum velocity:}
\]

\[T_2 = \frac{1}{2} I_B \dot{\theta}_m^2 \quad \text{but} \quad I_B = 2 \left( \frac{1}{3} ml^2 \right)\]

Thus,

\[T_2 = \frac{1}{2} \left( \frac{2}{3} ml^2 \right) \dot{\theta}_m^2\]

\[V_2 = -2mg \left( \frac{l}{2} \cos \beta \right) = -mgl \cos \beta\]
PROBLEM 19.74 (Continued)

Conservation of energy.\[ T_1 + V_1 = T_2 + V_2 \]

\[ -mg/l \cos \beta + \frac{1}{2} mgl \cos \beta \theta_m^2 = \frac{1}{3} ml^2 \dot{\theta}_m^2 - mgl \cos \beta \]

Setting \( \dot{\theta}_m = \theta_m \omega_n \), \[ \frac{1}{2} mgl \cos \beta \theta_m^2 = \frac{1}{3} ml^2 \theta_m^2 \omega_n^2 \]

\[ \omega_n^2 = \frac{3g}{2l} \cos \beta \quad \tau = \frac{2\pi}{\omega_n} = 2\pi \sqrt{\frac{2l}{3g \cos \beta}} \tag{1} \]

But for \( \beta = 0 \), \[ \tau_0 = 2\pi \sqrt{\frac{2l}{3g}} \]

For \( \tau = 2\tau_0 \), \[ 2\pi \sqrt{\frac{2l}{3g \cos \beta}} = 2 \left( 2\pi \sqrt{\frac{2l}{3g}} \right) \]

Squaring and reducing, \[ \frac{1}{\cos \beta} = 4 \quad \cos \beta = \frac{1}{4} \quad \beta = 75.5^\circ \leftarrow \]
PROBLEM 19.75

The inner rim of an 40 kg flywheel is placed on a knife edge, and the period of its small oscillations is found to be 1.26 s. Determine the centroidal moment of inertia of the flywheel.

SOLUTION

Datum at ①:

**Position ①**

\[ T_1 = \frac{1}{2} I_0 \dot{\theta}_m^2 \quad V_1 = 0 \]

**Position ②**

\[ T_2 = 0 \quad V_2 = mgh \]

\[ h = r(1 - \cos \theta_m) = r \sin^2 \frac{\theta_m}{2} = \frac{r \theta_m^2}{2} \]

\[ V_2 = mgr \frac{\theta_m^2}{2} \]

Conservation of energy:

\[ T_1 + V_1 = T_2 + V_2: \quad \frac{1}{2} I_0 \dot{\theta}_m^2 + 0 = 0 + mgr \frac{\theta_m^2}{2} \]

For simple harmonic motion, \[ \dot{\theta}_m = \omega_n \theta_m \]

\[ I_0 \omega_n^2 \theta_m^2 = mgr \theta_m^2 \quad \omega_n^2 = \frac{mgr}{J_0} \quad \frac{\omega_n^2}{2} = \frac{(4\pi^2)}{mgr} \]

Moment of inertia.

\[ I_0 = \bar{T} + m r^2 \quad \bar{T} + m r^2 = \frac{(\tau_n^2)(mgr)}{4\pi^2} \]

\[ \bar{T} = \frac{(\tau_n^2)(mgr)}{4\pi^2} - m r^2 = \frac{(1.26)^2 (9.81)}{4\pi^2} \left( \frac{0.35}{2} \right)^2 - (40) \left( \frac{0.35}{2} \right)^2 \]

\[ \bar{T} = 2.7615 - 1.225 \quad \bar{T} = 1.537 \text{ kg\cdotm}^2 \]
PROBLEM 19.76

A connecting rod is supported by a knife edge at Point A; the period of its small oscillations is observed to be 1.03 s. Knowing that the distance \( r_a \) is 150 mm, determine the centroidal radius of gyration of the connecting rod.

\[
\begin{align*}
T_1 &= 0, \quad V_1 = mgr_a(1 - \cos \theta_m) = \frac{1}{2} mgr_a \theta_m^2 \\
T_2 &= \frac{1}{2} mv_G^2 + \frac{1}{2} \dot{r}_m^2 = \frac{1}{2} m \dot{r}_m^2 + \frac{1}{2} \frac{gr_a^2 \theta_m^2}{r_a^2 + k^2} \\
&= \frac{1}{2} m (r_a^2 + k^2) \dot{\theta}_m^2 \\
V_2 &= 0 \\
\text{For simple harmonic motion,} \quad \dot{\theta}_m = \omega_m \theta_m \\
\text{Conservation of energy.} \quad T_1 + V_1 = T_2 + V_2
\end{align*}
\]

\[
\begin{align*}
0 + \frac{1}{2} mgr_a \theta_m^2 &= \frac{1}{2} m (r_a^2 + k^2) \omega_m^2 + 0 \\
\omega_m^2 &= \frac{gr_a}{r_a^2 + k^2} \quad \text{or} \quad k^2 = \frac{gr_a}{\omega_m^2} - r_a^2 \\
\end{align*}
\]

Data:

\[
\begin{align*}
\tau_n &= 1.03 \text{ s} \quad \omega_n = \frac{2\pi}{\tau_n} = \frac{2\pi}{1.03} = 6.1002 \text{ rad/s} \\
r_a &= 150 \text{ mm} = 0.15 \text{ m} \quad g = 9.81 \text{ m/s}^2 \\
k^2 &= \frac{(9.81)(0.15)}{(6.1002)^2} - (0.15)^2 = 0.039543 - 0.0225 = 0.017043 \text{ m}^2 \\
k &= 0.13055 \text{ m} \quad k = 130.6 \text{ mm}
\end{align*}
\]
PROBLEM 19.77

The rod $ABC$ of total mass $m$ is bent as shown and is supported in a vertical plane by a pin at $B$ and a spring of constant $k$ at $C$. If end $C$ is given a small displacement and released, determine the frequency of the resulting motion in terms of $m$, $L$, and $k$.

SOLUTION

Position ①

$$T_1 = \frac{1}{2} I_B \dot{\theta}_m^2 \quad V_1 = \frac{1}{2} k (\delta_{ST})^2$$

Position ②

$$T_2 = 0 \quad V_2 = -\frac{m}{2} gh - \frac{m}{2} g \frac{L}{2} \sin \theta_m + \frac{1}{2} k(L \theta_m + \delta_{ST})^2$$

$$h = \frac{L}{2} (1 - \cos \theta_m) = 2 \frac{L}{2} \sin^2 \theta_m$$

$$h = \frac{\dot{\theta}_m^2 mL}{4} \quad V_2 = -\frac{m}{2} g \frac{L}{4} \dot{\theta}_m^2 - \frac{m}{2} g \frac{L}{2} \dot{\theta}_m + \frac{1}{2} k(L \theta_m + \delta_{ST})^2$$

$$\sin \theta_m = \theta_m$$

Conservation of energy.

$$T_1 + V_1 = T_2 + V_2$$

$$\frac{1}{2} I_B \dot{\theta}_m^2 + \frac{1}{2} k \delta_{ST}^2 = 0 - \frac{m}{2} g \frac{L \dot{\theta}_m^2}{4} - \frac{mgL}{2} \dot{\theta}_m + \frac{1}{2} k \left[ L^2 \dot{\theta}_m^2 + \delta_{ST}^2 + 2L \delta_{ST} \theta_m \right]$$

When the rod is in equilibrium,

$$\sum M_B = 0 = \frac{m}{2} g \frac{L}{2} - kL \delta_{ST}$$
PROBLEM 19.77 (Continued)

Substituting Eq. (2) into Eq. (1)

\[ I_B \ddot{\theta}_m^2 = \left( kL^2 - \frac{mgL}{4} \right) \dot{\theta}_m^2 \quad \dot{\theta}_m = \omega_n \theta_m \]

\[ I_B \omega_n^2 \theta_m^2 = \left( kL^2 - \frac{mgL}{4} \right) \dot{\theta}_m^2 \]

\[ \omega_n^2 = \frac{kL^2 - \frac{mgL}{4}}{I_B} \]

\[ I_B = 2 \left( \frac{1}{3} \frac{mL^2}{2} \right) = \frac{mL^2}{3} \]

\[ \omega_n^2 = \frac{kL^2 - \frac{mgL}{4}}{\frac{mL^2}{3}} = \frac{3k}{m} - \frac{3g}{4L} \]

Frequency of oscillation.

\[ f_n = \frac{\omega_n}{2\pi} = \frac{\sqrt{3}}{2\pi} \sqrt{\frac{k}{m} - \frac{g}{4L}} \]
PROBLEM 19.78

A 7.5 kg uniform cylinder can roll without sliding on an incline and is attached to a spring $AB$ as shown. If the center of the cylinder is moved 10 mm down the incline and released, determine (a) the period of vibration, (b) the maximum velocity of the center of the cylinder.

SOLUTION

(a) Position ①

$$T_1 = 0 \quad V_1 = \frac{1}{2} k(\delta_{ST} + r\theta_m)^2$$

Position ②

$$T_2 = \frac{1}{2} J\dot{\theta}_m^2 + \frac{1}{2} mV_m^2$$

$$V_2 = mgh + \frac{1}{2} k(\delta_{ST}^2)$$

Conservation of energy,

$$T_1 + V_1 = T_2 + V_2: \quad 0 + \frac{1}{2} k(\delta_{ST} + r\theta_m)^2 = \frac{1}{2} J\dot{\theta}_m^2 + \frac{1}{2} mV_m^2 + mgh + \frac{1}{2} k(\delta_{ST})^2$$

$$k\delta_{ST}^2 + 2k\delta_{ST}r\theta_m + kr^2\theta_m^2 = J\dot{\theta}_m^2 + mV_m^2 + 2mgh + k\delta_{ST}^2 \quad (1)$$

When the disk is in equilibrium,

$$\sum M_c = 0 = mg \sin \beta r - k\delta_{ST}r$$

Also,

$$h = r \sin \beta \theta_m$$

Thus,

$$mgh - k\delta_{ST}r = 0 \quad (2)$$
PROBLEM 19.78 (Continued)

Substituting Eq. (2) into Eq. (1)

\[ kr^2 \dot{\theta}_m^2 = J \dot{\theta}_m^2 + m \ddot{v}_n \]
\[ \dot{\theta}_m = \omega_n \theta_m \quad v_m = r \dot{\theta}_m = r \omega_n \theta_m \]

\[ kr^2 \dot{\theta}_m^2 = (J + mr^2) \dot{\theta}_m^2 \omega_n^2 \]

\[ \omega_n^2 = \frac{kr^2}{J + mr^2} \quad J = \frac{1}{2} mr^2 \]

\[ \omega_n^2 = \frac{1}{2} \frac{kr^2}{mr^2 + mr^2} = \frac{2}{3} \frac{k}{m} \]

\[ \tau_n = \frac{2 \pi}{\omega_n} = \frac{2 \pi}{\sqrt{\frac{2}{3} \left( \frac{900 \text{ N/m}}{7.5 \text{ kg}} \right)}} = \frac{2 \pi}{\sqrt{80}} \approx 0.702 \text{ s} \]

(b) Maximum velocity.

\[ v_m = r \dot{\theta}_m \]
\[ \theta_m = \theta_m \omega_n \]
\[ v_m = r \dot{\theta}_m \omega_n \quad r \dot{\theta}_m = \frac{10}{1000} = 0.01 \text{ m} \]

\[ v_m = (0.01 \text{ m/s}) \sqrt{80} \approx 0.0894 \text{ m/s} \]
**PROBLEM 19.79**

Two uniform rods, each of weight \( m = 600 \, \text{g} \) and length \( l = 200 \, \text{mm} \), are welded together to form the assembly shown. Knowing that the constant of each spring is \( k = 120 \, \text{N/m} \) and that end \( A \) is given a small displacement and released, determine the frequency of the resulting motion.

**SOLUTION**

**Mass and moment of inertia of one rod.** \( m = 600 \, \text{g} = 0.6 \, \text{kg} \)

\[
I = \frac{1}{12} ml^2 = \frac{1}{12} (0.6)(0.2)^2 = 2 \times 10^{-3} \, \text{kg} \cdot \text{m}^2
\]

**Approximation.**

\[
\sin \theta_m = \tan \theta_m = \theta_m
\]

\[
1 - \cos \theta_m = 2 \sin^2 \frac{\theta_m}{2} = \frac{1}{2} \theta_m^2
\]

**Spring constant:** \( k = 120 \, \text{N/m} \)

**Position \( \mathbb{O} \)**

\[
T_1 = 2 \left( \frac{1}{2} I \dot{\theta}_m^2 \right) + \frac{1}{2} m \left( \frac{l}{2} \dot{\theta}_m \right)^2
\]

\[
= (2) \left( \frac{1}{2} \right) (2 \times 10^{-3}) \dot{\theta}_m^2 + \left( \frac{1}{2} \right) (0.6)(0.1 \dot{\theta}_m)^2
\]

\[
= 5 \times 10^{-3} \dot{\theta}_m^2
\]

\( V_1 = 0 \)
PROBLEM 19.79 (Continued)

**Position**

\[ T_2 = 0 \]
\[ V_2 = -\frac{Wl}{2} (1 - \cos \theta_m) + 2 \left( \frac{1}{2} \right) k \left( \frac{1}{2} \theta_m \right)^2 \]
\[ = -\frac{(0.6)(9.81)(0.2)}{2} \left( \frac{1}{2} \theta_m \right)^2 + (2) \left( \frac{1}{2} \right) (120) \left( \frac{0.2}{2} \theta_m \right)^2 \]
\[ = 1.4943 \theta_m^2 \]

**Conservation of energy.**

\[ T_1 + V_1 = T_2 + V_2 \]
\[ 5 \times 10^{-3} \dot{\theta}_m^2 + 0 = 0 + 1.4943 \theta_m^2 \]
\[ \dot{\theta}_m = 17.2876 \theta_m \]

**Simple harmonic motion.**

\[ \dot{\theta}_m = \omega_n \theta_m \]
\[ \omega_n = 17.2876 \text{ rad/s} \]

**Frequency.**

\[ f_n = \frac{\omega_n}{2\pi} \]
\[ f_n = 2.75 \text{ Hz} \]
PROBLEM 19.80

A slender 8-kg rod $AB$ of length $l = 600$ mm is connected to two collars of negligible mass. Collar $A$ is attached to a spring of constant $k = 1.2 \text{kN/m}$ and can slide on a vertical rod, while collar $B$ can slide freely on a horizontal rod. Knowing that the system is in equilibrium and that $\theta = 40^\circ$, determine the period of vibration if collar $B$ is given a small displacement and released.

SOLUTION

Vertical rod.

$$ y = l \sin \theta $$
$$ \delta y = l \cos \theta \delta \theta $$
$$ \delta \dot{x} = -l \sin \theta \delta \dot{\theta} $$

$\bar{y} = \frac{y}{2}$ \quad $\bar{x} = \frac{x}{2}$

**Position $\text{①}$ (Maximum velocity, $\delta \theta m$)**

$$ T_1 = \frac{1}{2} I (\delta \theta)_m^2 + \frac{1}{2} m \left[ (\delta \dot{x}_m)^2 + (\delta \bar{y})^2 \right] $$

$$ T_1 = \frac{1}{2} \left( \frac{1}{12} ml^2 \right) (\delta \theta)_m^2 + m \left[ \left( \frac{l}{2} \sin \theta \right)^2 + \left( \frac{l}{2} \cos \theta \right)^2 \right] (\delta \bar{y})^2 $$

$$ T_1 = \frac{1}{2} ml^2 \left[ \frac{1}{12} + \frac{1}{4} \right] (\delta \theta_m)^2 = \frac{1}{2} ml^2 \frac{3}{1} (\delta \theta_m)^2 $$

$$ V_1 = \frac{1}{2} k (\delta \theta_m)^2 + mg \bar{y} $$

**Position $\text{②}$ (Zero velocity, maximum $\delta \theta_m$)**

$$ T_2 = 0 $$

$$ V_2 = \frac{1}{2} k (\delta \bar{y} + \delta \bar{y}_m)^2 + mg(\bar{y} - \delta \bar{y}_m) $$

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PROBLEM 19.80 (Continued)

Conservation of energy. \[ T_1 + V_1 = T_2 + V_2 : \]
\[
\frac{1}{2} ml^2 + \frac{1}{3} (\delta \dot{\theta}_m)^2 + \frac{1}{2} k \delta_{ST}^2 + mg\overline{y} = 0 + \frac{1}{2} (\delta y + \delta_{ST})^2 + mg(\overline{y} - \delta_{ST}) \tag{1}
\]
\[
m l^2 \frac{1}{3} (\delta \dot{\theta}_m)^2 + k \delta_{ST}^2 + mg\overline{y} = k(\delta y^2 + 2\delta y \delta_{ST} + \delta_{ST}^2) + mg(\overline{y} - \delta_{ST})
\]

But when the rod is in equilibrium,
\[
\sum \Sigma M_g = mg \frac{x}{2} - k \delta_{ST} x = 0 \quad mg = 2k \delta_{ST} \tag{2}
\]

Substituting Eq. (2) into Eq. (1),
\[
m l^2 \frac{1}{3} (\delta \dot{\theta}_m)^2 = k \delta y_m^2 \quad \delta y_m = l \cos \theta \delta \theta_m
\]
\[
m l^2 \frac{1}{3} (\delta \dot{\theta}_m)^2 = kl^2 \cos^2 \theta (\delta \theta_m)^2
\]

For simple harmonic motion,
\[
\delta \theta = \delta \theta_m \sin(\omega_m t + \phi)
\]
\[
\delta \dot{\theta}_m = \omega_m \delta \theta_m
\]
\[
\frac{1}{3} m (\delta \theta_m)^2 \omega_n^2 = k \cos^2 \theta (\delta \theta_m)^2
\]
\[
\omega_n^2 = \frac{k}{m} \cos^2 \theta = \frac{3(1200 \text{ N/m})}{8 \text{ kg}} \cos^2 40^\circ
\]
\[
\omega_n^2 = 264.07
\]

Period of vibration.
\[
\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{264.07}} \quad \tau_n = 0.387 \text{ s} \quad \blacktriangle
\]
PROBLEM 19.81

A slender rod \( AB \) of length \( l = 600 \text{ mm} \) and negligible mass is connected to two collars, each of mass 8 kg. Collar \( A \) is attached to a spring of constant \( k = 1.2 \text{ kN/m} \) and can slide on a vertical rod, while collar \( B \) can slide freely on a horizontal rod. Knowing that the system is in equilibrium and that \( \theta = 40^\circ \), determine the period of vibration if collar \( A \) is given a small displacement and released.

SOLUTION

Horizontal rod.

\[
Y = l \sin \theta \\
\delta Y = l \cos \theta \delta \theta \\
\delta x = l \cos \theta \\
x = l \cos \theta \\
\delta Y = -l \sin \theta \delta \theta \\
\delta x = -l \sin \theta \delta \theta
\]

Position \( \circ \) (Maximum velocity, \( \delta \dot{\theta} \))

\[
T_1 = \frac{1}{2} m(\delta \dot{y}_m)^2 + \frac{1}{2} m(\delta \dot{x}_m)^2 \\
T_1 = \frac{1}{2} m[(l \cos \theta)^2 + (l \sin \theta)^2](\delta \dot{\theta}_m)^2 \\
T_1 = \frac{1}{2} ml^2(\delta \dot{\theta}_m)^2 \\
V_1 = 0
\]

Position \( \odot \) (Zero velocity, maximum \( \delta \theta \))

\[
T_2 = 0 \\
V_2 = \frac{1}{2} k \delta y_m^2
\]

Conservation of energy.

\[
T_1 + V_1 = T_2 + V_2 \\
\frac{1}{2} ml^2(\delta \dot{\theta}_m)^2 + 0 = \frac{1}{2} k \delta y_m^2 \\
\delta y_m = l \cos \theta \delta \theta_m \\
ml^2(\delta \dot{\theta}_m)^2 = kl^2 \cos^2 \theta(\delta \theta_m)^2
\]
PROBLEM 19.81 (Continued)

Simple harmonic motion.

\[ \delta \theta = \delta \theta_m \sin(\omega_n t + \phi) \]
\[ \delta \dot{\theta}_m = \delta \theta_m \omega_n \]
\[ ml^2 (\delta \theta_m)^2 \omega_n^2 = kl^2 \cos^2 \theta (\delta \theta_m)^2 \]
\[ \omega_n^2 = \frac{k}{m} \cos^2 \theta \]
\[ \omega_n^2 = \frac{1200 \text{ N/m}}{8 \text{ kg}} \cos^2 40 = 88.02 \text{ s}^{-2} \]

Period of vibration.

\[ \tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{88.02}} = 0.6697 \text{ s} \]

\[ \tau_n = 0.670 \text{ s} \]
PROBLEM 19.82

A 3-kg slender rod $AB$ is bolted to a 5-kg uniform disk. A spring of constant 280 N/m is attached to the disk and is unstretched in the position shown. If end $B$ of the rod is given a small displacement and released, determine the period of vibration of the system.

SOLUTION

Position ①

$$T_1 = \frac{1}{2} J_{\text{disk}} \dot{\theta}^2 + \frac{1}{2} (J_{\text{rod}}) \dot{\theta}^2$$

$$V_1 = 0$$

$$J_{\text{disk}} = \frac{1}{2} m_0 r^2$$

$$(J_{\text{rod}}) = \frac{1}{3} m_{AB} l^2$$

Position ②

$$T_2 = 0$$

$$V_2 = \frac{1}{2} k (r \theta_m)^2 + m_{AB} g \frac{l}{2} (1 - \cos \theta_m)$$

$$1 - \cos \theta_m = 2 \sin^2 \frac{\theta_m}{2} = \frac{\theta_m^2}{2}$$

$$V_2 = \frac{1}{2} k r^2 \dot{\theta}^2 + m_{AB} g \frac{l}{2} \dot{\theta}_m^2$$
PROBLEM 19.82 (Continued)

Conservation of energy.

\[ T_1 + V_1 = T_2 + V_2: \quad \frac{1}{2} \left( \frac{1}{2} m_0 r^2 + \frac{1}{3} m_{AB} l^2 \right) \dot{\theta}_m^2 + 0 = 0 + \frac{1}{2} k r^2 \dot{\theta}_m^2 + \frac{1}{2} m_{AB} g \frac{l}{2} \dot{\theta}_m^2 \]

For simple harmonic motion,

\[ \dot{\theta}_m = \omega_m \theta_m \]

\[ \left( \frac{1}{2} m_0 r^2 + \frac{1}{3} m_{AB} l^2 \right) \omega_m^2 \theta_m^2 = \left( k r^2 + m_{AB} g \frac{l}{2} \right) \theta_m^2 \]

\[ \omega_m^2 = \frac{k r^2 + m_{AB} g l}{\frac{1}{2} m_0 r^2 + \frac{1}{3} m_{AB} l^2} \]

\[ \omega_n^2 = \frac{(280 \text{ N/m})(0.08 \text{ m})^2 + (3 \text{ kg})(9.81 \text{ m/s}^2)(0.3 \text{ m})}{\frac{1}{2}(5 \text{ kg})(0.08 \text{ m})^2 + \frac{1}{3}(3 \text{ kg})(0.300 \text{ m})^2} \]

\[ \omega_n^2 = \frac{6.207}{0.106} = 58.55 \]

Period of vibration.

\[ \tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{58.55}} \]

\[ \tau_n = 0.821 \text{ s} \]

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PROBLEM 19.83

A 400-g sphere $A$ and a 300-g sphere $C$ are attached to the ends of a 600-g rod $AC$, which can rotate in a vertical plane about an axis at $B$. Determine the period of small oscillations of the rod.

SOLUTION

Position 1

\[
T_1 = \frac{1}{2} m_A (0.125)^2 + \frac{1}{2} m_C (0.2 \theta_m)^2 + \frac{1}{2} m_{AC} (0.0375 \theta_m)^2 + \frac{1}{2} T_{AC} \theta_m^2
\]

\[
T_{AC} = \frac{1}{12} m_{AC} (0.325)^2
\]

\[
T_1 = \frac{1}{2} \left[ 0.4(0.125)^2 + 0.3(0.2)^2 + 0.6(0.0375)^2 + \frac{1}{12} (0.6)(0.325)^2 \right] \theta_m^2
\]

\[
T_1 = \frac{1}{2} (0.024375) \theta_m^2 \text{ (N·m)}
\]

\[
V_1 = 0
\]
PROBLEM 19.83 (Continued)

**Position**

\[ T_2 = 0 \]
\[ V_2 = -W_A 0.125(1 - \cos \theta_m) + W_C 0.2(1 - \cos \theta_m) + W_{AC} 0.0375(1 - \cos \theta_m) \]

\[ 1 - \cos \theta_m = 2 \sin^2 \frac{\theta_m}{2} = \frac{\theta_m^2}{2} \]
\[ V_2 = \left[ -(0.4)(9.81)(0.125) + (0.3)(9.81)(0.2) + (0.6)(9.81)(0.0375) \right] \frac{\theta_m^2}{2} \text{ (N} \cdot \text{m)} \]
\[ V_2 = \frac{0.318825\theta_m^2}{2} \]

**Conservation of energy.**
\[ T_1 + V_1 = T_2 + V_2^2 \]
\[ \frac{1}{2}(0.024375)|\theta_m|^2 + 0 = 0 + \frac{0.318825}{2}\theta_m^2 \]

**Simple harmonic motion.**
\[ \ddot{\theta}_m = \omega_n \theta_m \]
\[ \omega_n^2 = \frac{0.318825}{0.024375} = 13.08 \]
\[ \tau_n = \frac{2\pi}{\sqrt{\omega_n}} = \frac{2\pi}{\sqrt{13.08}} \]
\[ \tau_n = 1.737 \text{ s} \]
PROBLEM 19.84

Three identical rods are connected as shown. If \( b = \frac{3}{4} l \), determine the frequency of small oscillations of the system.

SOLUTION

\( l = \) length of each rod
\( m = \) mass of each rod

Kinematics:
\[
\ddot{v}_m = \frac{l}{2} \dot{\theta}_m
\]
\[
(\ddot{v}_{BE})_m = b \dot{\theta}_m
\]

Position ①

\[
T_1 = 0
\]
\[
V_1 = -2mg \frac{l}{2} \cos \theta_m - mb \cos \theta_m
\]
\[
V_1 = -mg(l + b) \cos \theta_m
\]

Position ②

\[
V_2 = -2mg \frac{l}{2} - mgb
\]
\[
= -mg(l + b)
\]
\[
T_2 = 2 \left[ \frac{1}{2} I \ddot{\theta}_m^2 + \frac{1}{2} m \dot{v}_m^2 \right] + \frac{1}{2} m(\ddot{v}_{BE})^2_m
\]
\[
= \frac{1}{12} ml^2 \ddot{\theta}_m^2 + m \left( \frac{l}{2} \dot{\theta}_m \right)^2 + \frac{1}{2} m(b \dot{\theta}_m)^2
\]
\[
T_2 = \left( \frac{1}{3} l^2 + \frac{1}{2} b^2 \right) m \ddot{\theta}_m^2
\]
Problem 19.84 (Continued)

Conservation of energy. \[ T_1 + V_1 = T_2 + V_2: \quad 0 - mg(l + b)\cos \theta_m = \left( \frac{1}{3} l^2 + \frac{1}{2} b^2 \right) \ddot{\theta}_m - mg(l + b) \]

\[ mg(l + b)(1 - \cos \theta_m) = \left( \frac{1}{3} l^2 + \frac{1}{2} b^2 \right) \ddot{\theta}_m \]

For small oscillations, \[ (1 - \cos \theta_m) = \frac{1}{2} \theta_m^2 \]

\[ \frac{1}{2} mg(l + b)\theta_m^2 = \left( \frac{1}{3} l^2 + \frac{1}{2} b^2 \right) m\ddot{\theta}_m \]

But for simple harmonic motion, \[ \dot{\theta}_m = \omega_n \theta_m: \quad \frac{1}{2} mg(l + b)\theta_m^2 = \left( \frac{1}{3} l^2 + \frac{1}{2} b^2 \right) m(\omega_n \theta_m)^2 \]

\[ \omega_n^2 = \frac{1}{2} g \frac{l + b}{3 l^2 + \frac{1}{2} b^2} \]

or \[ \omega_n^2 = 3 g \frac{l + b}{2 l^2 + 3 b^2} \]

For \( b = \frac{3}{4} l \), we have \[ \omega_n^2 = 3 g \frac{l + \frac{3}{4} l}{2 l^2 + 3 \left( \frac{3}{4} l \right)^2} \]

\[ = 3 g \frac{7 l / 4}{16 l^2} \]

\[ = 1.4237 \frac{g}{l} \]

\[ \omega_n = 1.1932 \sqrt{\frac{g}{l}} \]

\[ f_n = \frac{\omega_n}{2\pi} \]

\[ = \frac{1.1932}{2\pi} \sqrt{\frac{g}{l}} \quad f_n = 0.1899 \sqrt{\frac{g}{l}} \]
PROBLEM 19.85

An 800-g rod $AB$ is bolted to a 1.2-kg disk. A spring of constant $k = 12$ N/m is attached to the center of the disk at $A$ and to the wall at $C$. Knowing that the disk rolls without sliding, determine the period of small oscillations of the system.

SOLUTION

**Position ①**

$$T_1 = \frac{1}{2} (I_G)_{AB} \theta_m^2 + \frac{1}{2} m_{AB} \left( \frac{l}{2} - r \right)^2 \dot{\theta}_m + \frac{1}{2} (I_G)_{disk} \theta_m^2 + \frac{1}{2} m_{disk} r^2 \dot{\theta}_m^2$$

$$(I_G)_{AB} = \frac{1}{12} m l^2 = \frac{1}{12} (0.8)(0.6)^2 = 0.024 \text{ kg}\cdot\text{m}^2$$

$$m_{AB} \left( \frac{l}{2} - r \right)^2 = (0.8)(0.3 - 0.25)^2 = 0.002 \text{ kg}\cdot\text{m}^2$$

$$(I_G)_{disk} = \frac{1}{2} m_{disk} r^2 = \frac{1}{2} (1.2)(0.25)^2 = 0.0375 \text{ kg}\cdot\text{m}^2$$

$$m_{disk} r^2 = 1.2(0.25)^2 = 0.0750 \text{ kg}\cdot\text{m}^2$$

$$T_1 = \frac{1}{2} [0.024 + 0.002 + 0.0375 + 0.0750] \dot{\theta}_m^2$$

$$T_1 = \frac{1}{2} [0.1385] \dot{\theta}_m^2$$

$$V_1 = 0$$

**Position ②**

$$T_2 = 0$$

$$V_2 = \frac{1}{2} k(r \theta_m)^2 + m_{AB} g \frac{l}{2} (1 - \cos \theta_m)$$

$$1 - \cos \theta_m = 2 \sin^2 \frac{\theta_m}{2} = \frac{\theta_m^2}{2} \quad \text{(small angles)}$$

$$V_2 = \frac{1}{2} (12 \text{ N/m})(0.25 \text{ m})^2 \theta_m^2 + (8 \text{ kg})(9.81 \text{ m/s}^2) \left( \frac{0.6 \text{ m}}{2} \right) \frac{\theta_m^2}{2}$$
PROBLEM 19.85 (Continued)

\[ V_2 = \frac{1}{2} [0.750 + 2.354] \theta_m^2 \]
\[ = \frac{1}{2} (3.104) \theta_m^2 \text{ N} \cdot \text{m} \]

\[ T_1 + V_1 = T_2 + V_2 \]
\[ \dot{\theta}_m^2 = \omega_n^2 \theta_m^2 \]
\[ \frac{1}{2} (0.1385) \theta_m^2 \omega_n^2 + 0 = 0 + \frac{1}{2} (3.104) \theta_m^2 \]
\[ \omega_n^2 = \frac{(3.104 \text{ N} \cdot \text{m})}{(0.1385 \text{ kg} \cdot \text{m}^2)} \]
\[ = 22.41 \text{ s}^{-2} \]
\[ \tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{22.41}} = 1.327 \text{ s} \]

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PROBLEM 19.86

Two uniform rods $AB$ and $CD$, each of length $l$ and mass $m$, are attached to gears as shown. Knowing that the mass of gear $C$ is $m$ and that the mass of gear $A$ is $4m$, determine the period of small oscillations of the system.

SOLUTION

Kinematics:

\[
2r\theta_A = r\theta_C \\
2\theta_A = \theta_C \\
2\dot{\theta}_A = \dot{\theta}_C
\]

Let

\[
\theta_A = \theta_m \\
2\theta_m = (\theta_C)_m \\
2\dot{\theta}_m = (\dot{\theta}_C)_m
\]

Position ①

\[
T_1 = \frac{1}{2} T_A \dot{\theta}_m^2 + \frac{1}{2} T_C (2\dot{\theta}_m)^2 + \frac{1}{2} T_{AB} \dot{\theta}_m^2 + \frac{1}{2} T_{CD} (2\dot{\theta}_m)^2 \\
+ \frac{1}{2} m_{AB} \left( \frac{l}{2} \dot{\theta}_m \right)^2 + \frac{1}{2} m_{CD} \left( \frac{l}{2} \dot{\theta}_m \right)^2
\]

\[
T_A = \frac{1}{2} (4m)(2r)^2 = 8mr^2 \\
T_C = \frac{1}{2} (m)(r)^2 = \frac{1}{2} mr^2 \\
T_{AB} = \frac{1}{12} ml^2 \\
T_{CD} = \frac{1}{12} ml^2
\]

\[
T_1 = \frac{1}{2} m \left[ 8r^2 + \left( \frac{r^2}{2} \right) 4 + \frac{l^2}{12} + \frac{l^2}{3} + \frac{l^2}{4} + l^2 \right]
\]

\[
T_1 = \frac{1}{2} m \left[ 10r^2 + \frac{l^2}{3} \right] \dot{\theta}_m^2
\]

\[
V_1 = 0
\]
PROBLEM 19.86 (Continued)

Position \( T_1 = 0 \)

\[ V_1 = mg \frac{l}{2} (1 - \cos \theta_m) + \frac{mgl}{2} (1 - \cos 4\theta_m) \]

For small angles,

\( 1 - \cos \theta_m = 2 \sin^2 \frac{\theta_m}{2} = \frac{\theta_m^2}{2} \)

\( 1 - \cos 4\theta_m = 2 \sin^2 2\theta_m = 2\theta_m^2 \)

\[ V_1 = \frac{1}{2} mgl \left( \frac{\theta_m^2}{2} + 2\theta_m^2 \right) = \frac{1}{2} mgl \frac{5\theta_m^2}{2} \]

\[ T_1 + V_1 = T_2 + V_2 \quad \dot{\theta}_m^2 = \omega_n^2 \theta_m^2 \]

\[ \frac{1}{2} m \left[ 10r^2 + \frac{5}{3} l^2 \right] \omega_n^2 \theta_m^2 + 0 = 0 + \frac{1}{2} mgl \frac{5\theta_m^2}{2} \]

\[ \omega_n^2 = \frac{\frac{5}{3} gl}{10r^2 + \frac{5}{3} l^2} \]

\[ = \frac{3gl}{12r^2 + 2l^2} \]

\[ \tau_n = \frac{2\pi}{\sqrt{\omega_n}} = 2\pi \sqrt{\frac{12r^2 + 2l^2}{3gl}} \]
**PROBLEM 19.87**

Two uniform rods $AB$ and $CD$, each of length $l$ and mass $m$, are attached to gears as shown. Knowing that the mass of gear $C$ is $m$ and that the mass of gear $A$ is $4m$, determine the period of small oscillations of the system.

**SOLUTION**

Kinematics:

\[
2r\theta_A = r\theta_C, \quad 2\theta_A = \theta_C, \quad 2\dot{\theta}_A = \dot{\theta}_C
\]

Let

\[
\theta_A = \theta_m, \quad 2\theta_m = (\theta_C)_m, \quad 2\dot{\theta}_m = (\dot{\theta}_C)_m
\]

Position

\[
T_1 = \frac{1}{2} T_A \dot{\theta}_m^2 + \frac{1}{2} T_C (2\dot{\theta}_m)^2 + \frac{1}{2} T_{AB} \dot{\theta}_m^2 + \frac{1}{2} T_{CD} (2\dot{\theta}_m)^2 + \frac{1}{2} m_{AB} \left(\frac{l}{2} \dot{\theta}_m\right)^2 + \frac{1}{2} m_{CD} \left(\frac{l}{2} 2\dot{\theta}_m\right)^2
\]

\[
T_A = \frac{1}{2} (4m)(2r)^2 = 8mr^2
\]

\[
T_C = \frac{1}{2} (m)(r^2) = \frac{1}{2} mr^2
\]

\[
T_{AB} = \frac{1}{12} ml^2 \quad T_{CD} = \frac{1}{12} ml^2
\]

\[
T_1 = \frac{1}{2} m \left[ 8r^2 + \left(\frac{r^2}{2}\right) 4 + \frac{l^2}{12} + \frac{l^2}{3} + \frac{l^2}{4} + l^2 \right] \dot{\theta}_m^2
\]

\[
T_1 = \frac{1}{2} m \left[ 10r^2 + \frac{5}{3} l^2 \right] \dot{\theta}_m^2 \quad V_i = 0
\]
PROBLEM 19.87 (Continued)

\[ T_2 = 0 \]
\[ V_2 = -mg l \frac{1}{2} (1 - \cos \theta_m) + \frac{mg \ell}{2} (1 - \cos 2\theta_m) \]

For small angles,
\[ 1 - \cos \theta_m = 2 \sin^2 \frac{\theta_m}{2} = \frac{\theta_m^2}{2} \]
\[ 1 - \cos 2\theta_m = 2 \sin^2 \theta_m = 2\theta_m^2 \]
\[ V_2 = -mg \frac{l \theta_m^2}{2} + \frac{mg \ell}{2} 2\theta_m^2 \]
\[ = \frac{1}{2} mg l^2 \theta_m^2 \]
\[ T_1 + V_1 = T_2 + V_2 \quad \dot{\theta}_m = \omega_n \theta_m \]
\[ \frac{1}{2} m \left[ 10r^2 + \frac{5}{3} l^2 \right] \theta_m^2 \omega_n^2 + 0 = 0 + \frac{1}{2} mg l \frac{3}{2} \theta_m^2 \]
\[ \omega_n^2 = \frac{\frac{1}{2} gl}{10r^2 + \frac{5}{3} l^2} \]
\[ = \frac{\frac{9}{2} gl}{60r^2 + 10l^2} \]
\[ \tau_n = \frac{2\pi}{\sqrt{\omega_n}} = 2\pi \sqrt{\frac{60r^2 + 10l^2}{9gl}} \]
PROBLEM 19.88

A 5-kg uniform rod $CD$ is welded at $C$ to a shaft of negligible mass which is welded to the centers of two 10-kg uniform disks $A$ and $B$. Knowing that the disks roll without sliding, determine the period of small oscillations of the system.

SOLUTION

$$W_A = W_B = W_{\text{disk}}$$

Position ①

$$T_1 = \frac{1}{2} (\bar{T}_A)_{\text{disk}} \theta_m^2 + \frac{1}{2} (2m_{\text{disk}})(r \dot{\theta}_m)^2$$

$$\bar{T}_A = \frac{1}{2} m_{\text{disk}} r^2 = \frac{1}{2} (2 \times 10)(0.5)^2 = 1.25 \text{ kg} \cdot \text{m}^2$$

$$T_{CD} = \frac{1}{12} m_{CD} l^2 = \frac{1}{12} (5)(1.5)^2 = 0.9375 \text{ kg} \cdot \text{m}^2$$

$$T_1 = \frac{1}{2} [2.5 + 5 + 0.9375 + 0.3125] \theta_m^2$$

Position ②

$$T_2 = 0$$

$$V_2 = W_{CD} \frac{l}{2} (1 - \cos \theta_m)$$
PROBLEM 19.88 (Continued)

Small angles:

\[ 1 - \cos \theta_m = 2 \sin^2 \frac{\theta_m}{2} = \frac{\theta_m^2}{2} \]

\[ V_2 = \frac{1}{2} W_{CD} \frac{\theta_m^2}{2} \]

\[ = \frac{1}{2} (5)(9.81)(1.5) \frac{\theta_m^2}{2} \]

\[ = \frac{1}{2} (36.7875) \theta_m^2 \]

Conservation of energy and simple harmonic motion:

\[ T_1 + V_1 = T_2 + V_2 \]

\[ \dot{\theta}_m = \omega_n \theta_m \]

\[ \frac{1}{2} (8.75) \omega_n^2 \theta_m^2 + 0 = \frac{1}{2} (36.7875) \theta_m^2 \]

\[ \omega_n^2 = \frac{36.7875}{8.75} = 4.2043 \]

Period of oscillations:

\[ \tau_n = \frac{2\pi}{\sqrt{\omega_n}} = \frac{2\pi}{\sqrt{4.2043}} \]

\[ \tau_n = 3.06 \, \text{s} \]

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PROBLEM 19.89

Four bars of the same mass $m$ and equal length $l$ are connected by pins at $A$, $B$, $C$, and $D$ and can move in a horizontal plane. The bars are attached to four springs of the same constant $k$ and are in equilibrium in the position shown ($\theta = 45^\circ$). Determine the period of vibration if corners $A$ and $C$ are given small and equal displacements toward each other and released.

SOLUTION

Kinematics:

- $BC: \quad x_G = \frac{l}{2} \cos \theta, \quad \delta x_G = -\frac{l}{2} \sin \theta \delta \theta$
- $\delta \dot{x}_G = -\frac{l}{2} \sin \theta \delta \theta$
- $y_G = \frac{l}{2} \sin \theta, \quad \delta y_G = \frac{l}{2} \cos \theta \delta \theta$
- $\delta \dot{y}_G = \frac{l}{2} \cos \theta \delta \theta$
- $x_C = l \cos \theta, \quad \delta x_C = -l \sin \theta \delta \theta$
- $\delta \dot{x}_C = -l \sin \theta \delta \theta$
- $x_B = l \sin \theta, \quad \delta x_B = l \cos \theta \delta \theta$
- $\delta \dot{x}_B = l \cos \theta \delta \theta$
- $y_C = 0$
- $y_B = l \sin \theta, \quad \delta y_B = l \cos \theta \delta \theta$
- $\delta \dot{y}_B = l \cos \theta \delta \theta$

The kinetic energy is the same for all four bars.
**PROBLEM 19.89 (Continued)**

**Position 1**

\[
T_1 = 4 \left[ \frac{1}{2} \bar{T} (\delta \theta_m)^2 + \frac{1}{2} m(\delta \dot{x}_G)_m^2 + (\delta \dot{y}_G)_m^2 \right]
\]

\[
\bar{T} = \frac{1}{12} ml^2
\]

\[
T_1 = 2 ml^2 \left[ \frac{1}{12} + \frac{1}{4} (\sin^2 \theta_m + \cos^2 \theta_m) \right] \delta \dot{\theta}_m^2
\]

\[
T_1 = \frac{2}{3} ml^2
\]

\[V_1 = 0\]

**Position 2**

\[T_2 = 0\]

\[
V_2 = (2) \frac{1}{2} k(\delta x_m)^2 + (2) \frac{1}{2} k(\delta y_m)^2
\]

\[
V_2 = k(\dot{x}^2 \sin^2 \theta + \dot{x}^2 \cos^2 \theta) \delta \dot{\theta}_m^2
\]

\[
= k \dot{x}^2 \delta \dot{\theta}_m^2
\]

Conservation of energy.

\[
T_1 + V_1 = T_2 + V_2
\]

\[
\frac{2}{3} ml^2 (\delta \dot{\theta}_m)^2 + 0 = 0 + k \dot{x}^2 (\delta \theta_m)^2
\]

Simple harmonic motion.

\[
\delta \dot{\theta}_m = \omega_n \delta \theta_m
\]

\[
\omega_n^2 = \left( \frac{3 k}{2 m} \right)
\]

Period of vibration.

\[
\tau_n = \frac{2\pi}{\omega_n}
\]

\[
\tau_n = 2\pi \sqrt{\frac{2m}{3k}}
\]
PROBLEM 19.90

The 10-kg rod $AB$ is attached to two 4-kg disks as shown. Knowing that the disks roll without sliding, determine the frequency of small oscillations of the system.

SOLUTION

<table>
<thead>
<tr>
<th>Position ②</th>
<th>Position ①</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r = 0.15$ m</td>
<td></td>
</tr>
</tbody>
</table>

Masses and moments of inertia.

$m_A = m_B = 4$ kg

$T_A = T_B = \frac{1}{2}m_A r_A^2 = \frac{1}{2}(4)(0.15)^2 = 0.045$ kg $\cdot$ m$^2$

$m_{AB} = 10$ kg

Kinematics:

$v_m = r_A \dot{\theta}_m = 0.15 \dot{\theta}_m$

$v_{AB} = 0.05 \dot{\theta}_m$

Position ① (Maximum displacement)

$T_1 = 0$

$V_1 = -W_{AB}(0.1 \cos \theta_m) = -(10)(9.81)(0.1 \cos \theta_m)$

$= -9.81 \cos \theta_m$

Position ② (Maximum speed)

$T_2 = \frac{1}{2}m_A v_m^2 + \frac{1}{2}T_A \dot{\theta}_m^2 + \frac{1}{2}m_B v_m^2 + \frac{1}{2}T_B \dot{\theta}_m^2 + \frac{1}{2}m_{AB} v_{AB}^2$

$= \frac{1}{2} \left[ (4)(0.15 \dot{\theta}_m)^2 + \frac{1}{2}(0.045)\dot{\theta}_m^2 \right] + \frac{1}{2}(10)(0.05\dot{\theta}_m)^2$

$= 0.1475 \dot{\theta}_m^2$

$V_2 = -W_{AB}(0.1) = -(10)(9.81)(0.1) = -9.81$
PROBLEM 19.90 (Continued)

Conservation of energy.

\[ T_1 + V_1 = T_2 + V_2 \]
\[-9.81 \cos \theta_m = 0.1475 \dot{\theta}_m^2 - 9.81 \]
\[ \dot{\theta}_m^2 = 66.5085(1 - \cos \theta_m) \]
\[ = 66.5085 \left( \frac{1}{2} \theta_m^2 \right) \]
\[ = 33.2542 \theta_m^2 \]
\[ \theta_m = 5.7666 \theta_m \]

Simple harmonic motion.

\[ \dot{\theta}_m = \omega_n \theta_m \]
\[ \omega_n = 5.7666 \text{ rad/s} \]

Frequency.

\[ f_n = \frac{\omega_n}{2\pi} = \frac{5.7666}{2\pi} \]
\[ f_n = 0.918 \text{ Hz} \]
PROBLEM 19.91

An inverted pendulum consisting of a sphere of mass \( m \) and a rigid bar \( ABC \) of length \( l \) and negligible weight is supported by a pin and bracket at \( C \). A spring of constant \( k \) is attached to the bar at \( B \) and is undeformed when the bar is in the vertical position shown. Determine \((a)\) the frequency of small oscillations, \((b)\) the smallest value of \( a \) for which these oscillations will occur.

SOLUTION

\((a)\) Position ① 

\[ T_1 = \frac{1}{2} m (l \dot{\theta}_m)^2 = \frac{1}{2} ml^2 \dot{\theta}_m^2 \]

\[ V_1 = 0 \]

Position ② 

\[ T_2 = 0 \]

\[ V_2 = \frac{1}{2} k (a \sin \theta_m)^2 - ml (1 - \cos \theta_m) \]

For small angles, 

\[ \sin \theta_m = \theta_m \]

\[ 1 - \cos \theta_m = 2 \sin^2 \frac{\theta_m}{2} = \frac{\theta_m^2}{2} \]

\[ V_2 = \frac{1}{2} k \theta_m^2 - ml \frac{\theta_m^2}{2} = \frac{1}{2} [k \theta_m^2 - ml \theta_m^2] \]

\[ T_1 + V_1 = T_2 + V_2 \]

\[ \frac{1}{2} ml^2 \dot{\theta}_m^2 + 0 = 0 + \frac{1}{2} [k \theta_m^2 - ml \theta_m^2] \]
PROBLEM 19.91 (Continued)

\[ \dot{\theta}_m = \omega_n \theta_m \]

\[ m = \frac{m}{g} \]

\[ \frac{m}{g} l^2 \omega_n^2 \theta_m^2 = ka^2 - ml \]

\[ \omega_n^2 = \frac{ka^2 - ml}{\left( \frac{m}{g} \right) l^2} \]

\[ = \frac{g}{l} \left[ \frac{ka^2}{ml} - 1 \right] \]

\[ f_n = 0 \]

\[ \frac{ka^2}{ml} - 1 > 0 \]

\[ a > \sqrt{\frac{ml}{k}} \]

\[ f_n = \frac{1}{2\pi} \omega_n = \frac{1}{2\pi} \sqrt{\frac{g}{l} \left( \frac{ka^2}{ml} - 1 \right)} \]
PROBLEM 19.92

For the inverted pendulum of Problem 19.91 and for given values of \( k, a, \) and \( l, \) it is observed that \( f = 1.5 \) Hz when \( m = 1 \) kg and that \( f = 0.8 \) Hz when \( m = 2 \) kg. Determine the largest value of \( W \) for which small oscillations will occur.

SOLUTION

See Solution to Problem 19.91 for the frequency in terms of \( W, k, a, \) and \( l, \)

\[
f_n = \frac{1}{2\pi} \sqrt{\frac{g}{l}} \left( \frac{ka^2}{Wl} - 1 \right)
\]

\( f_n = 1.5 \) Hz \( W = (1) (g) \) N \( 1.5 = \frac{1}{2\pi} \sqrt{\frac{g}{l}} \left( \frac{ka^2}{g} - 1 \right) \) \quad (1)

\( f_n = 0.8 \) Hz \( W = (2) (g) \) N \( 0.8 = \frac{1}{2\pi} \sqrt{\frac{g}{l}} \left( \frac{ka^2}{2gl} - 1 \right) \) \quad (2)

Dividing Eq. (1) by Eq. (2) and squaring,

\[
\left( \frac{1.5}{0.8} \right)^2 = \left( \frac{\frac{ka^2}{g} - 1}{\frac{ka^2}{2gl} - 1} \right) = \frac{2(ka^2 - gl)}{(ka^2 - 2gl)}
\]

\[
\frac{225}{64} (ka^2 - 2gl) = 2(ka^2 - gl)
\]

\[
ka^2 = 3.3196gl
\]

\[
f_n = \frac{1}{2\pi} \sqrt{\frac{g}{l}} \left( \frac{3.3196gl}{Wl} - 1 \right)
\]

\[
f_n = 0
\]

\[
\frac{3.3196g}{W} - 1 = 0
\]

\( W = 32.6 \) N \( ◊ \)
PROBLEM 19.93

A uniform rod of length \( L \) is supported by a ball-and-socket joint at \( A \) and by a vertical wire \( CD \). Derive an expression for the period of oscillation of the rod if end \( B \) is given a small horizontal displacement and then released.

SOLUTION

**Position \( \circlearrowleft \) (Maximum deflection)**

Looking from above:

Horizontal displacement of \( C \):

\[
x_C = b\theta_m
\]

\[
\phi_m = \frac{x_C}{h} = \frac{b}{h}\theta_m
\]

Looking from right:

\[
y_C = h(1 - \cos \phi_m) = \frac{1}{2} h\phi_m^2
\]

\[
y_C = \frac{1}{2} h \left( \frac{b}{h} \theta_m \right)^2 = \frac{1}{2} \frac{b^2}{h} \theta_m^2
\]

\[
y_m = y_G - y_C = \frac{AG}{AC} - y_C = \frac{1}{2} L \left( \frac{1}{2} \frac{b^2}{h} \theta_m^2 \right)
\]

\[
y_m = \frac{1}{4} \frac{bL}{h} \theta_m^2
\]

We have

\[
T_1 = 0
\]

\[
V_1 = mgy_m = \frac{1}{4} \frac{mgbL}{h} \theta_m^2
\]

**Position \( \circlearrowright \) (Maximum velocity)**

Looking from above:

\[
T_2 = \frac{1}{2} I \dot{\theta}_m^2 + \frac{1}{2} m\dot{v}_m^2
\]

\[
= \frac{1}{2} \left( \frac{1}{12} mL^2 \right) \dot{\theta}_m^2 + \frac{1}{2} m \left( \frac{L}{2} \dot{\theta} \right)^2
\]

\[
T_2 = \frac{1}{6} mL^2 \dot{\theta}_m^2
\]

\[
V_2 = 0
\]

**Conservation of energy.**

\[
T_1 + V_1 = T_2 + V_2: \quad 0 + \frac{1}{4} \frac{mgbL}{h} \theta_m^2 = \frac{1}{6} mL^2 \dot{\theta}_m^2
\]
PROBLEM 19.93 (Continued)

But for simple harmonic motion,

\[ \dot{\theta}_n = \omega_n \theta_n \]
\[ \frac{1}{4} \frac{mg bL}{n} \dot{\theta}_n^2 = \frac{1}{6} mL^2 (\omega_n \theta_n)^2 \]
\[ \omega_n^2 = \frac{3 bg}{2 hL} \]

Period of vibration.

\[ \tau_n = \frac{2\pi}{\omega_n} \]
\[ \tau_n = 2\pi \sqrt{\frac{2hL}{3bg}} \]
**PROBLEM 19.94**

A 2-kg uniform rod $ABC$ is supported by a pin at $B$ and is attached to a spring at $C$. It is connected at $A$ to a 2-kg block $DE$, which is attached to a spring and can roll without friction. Knowing that each spring can act in tension or compression, determine the frequency of small oscillations of the system when the rod is rotated through a small angle and released.

**SOLUTION**

\[
T_i = \frac{1}{2}(T_k)\dot{\theta}_m^2 + \frac{1}{2}(m_RC^2)\dot{\theta}_m + \frac{1}{2}m_E(\dot{x}_1)^2
\]

\[
T_k = \frac{1}{12}m_rl^2
\]

\[
= \frac{1}{12}(2 \text{ kg})(0.9 \text{ m})^2
\]

\[
= 0.135 \text{ kg} \cdot \text{m}^2
\]

\[
m_CC^2 = 2 \text{ kg}(0.15 \text{ m})^2 = 0.0225 \text{ kg} \cdot \text{m}^2
\]

\[
x_1 = 0.6\theta \quad m_E(\dot{x}_1)^2 = (2 \text{ kg})(0.6\dot{\theta}_m)^2
\]

\[
\dot{x}_1 = 0.6\dot{\theta} \quad m_E(\dot{x}_1)^2 = 0.72\dot{\theta}_m^2 \text{ m}^2
\]

\[
T_i = \frac{1}{2}[0.135 + 0.0225 + 0.72]\dot{\theta}_m^2
\]

\[
T_i = \frac{1}{2}(0.9)\dot{\theta}_m
\]

\[
V_i = 0
\]
PROBLEM 19.94 (Continued)

Position

\[ T_2 = 0 \]
\[ V_2 = \frac{1}{2} k(x_1)_m^2 + \frac{1}{2} k(x_2)_m^2 - m_0 g c (1 - \cos \theta_m) \]
\[ (x_1)_m = 0.6\theta_m \quad x_2 = 0.3\theta_m \]
\[ 1 - \cos \theta_m = 2 \sin^2 \frac{\theta}{2} = \theta_m^2 \]
\[ V_2 = \frac{1}{2} [(50 \text{ N/m})(0.6 \text{ m})^2 \theta_m^2 + (50 \text{ N/m})(0.3 \text{ m})^2 \theta_m^2 \]
\[ -(2 \text{ kg})(9.81 \text{ m/s}^2)(0.15)\theta_m^2 \]
\[ V_2 = \frac{1}{2} [18 + 4.5 - 2.943]\theta_m^2 \]
\[ V_2 = \frac{1}{2} (19.55)\theta_m^2 \]

Conservation of energy.

\[ T_i + V_i = T_2 + V_2: \quad \frac{1}{2} (0.9)\theta_m^2 + 0 = 0 + \frac{1}{2} (19.55)\theta_m^2 \]

Simple harmonic motion.

\[ \dot{\theta}_m = \omega_n^2 \theta_m^2 \quad (0.9)(\omega_n^2)\theta_m^2 = (19.55)\theta_m^2 \]
\[ \omega_n^2 = \frac{19.55}{0.9} = 21.73 \text{ s}^{-2} \]

Frequency of oscillations.

\[ f_n = \frac{\omega_n}{2\pi} = \frac{\sqrt{21.73}}{2\pi} \]
\[ f_n = 0.742 \text{ Hz} \]

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PROBLEM 19.95

A 750-g uniform arm $ABC$ is supported by a pin at $B$ and is attached to a spring at $A$. It is connected at $C$ to a 1.5-kg mass $m$ which is attached to a spring. Knowing that each spring can act in tension or compression, determine the frequency of small oscillations of the system when the weight is given a small vertical displacement and released.

SOLUTION

Position

$$k_C = 300 \text{ N/m}$$
$$k_A = 400 \text{ N/m}$$

$$T_1 = \frac{1}{2} T_{BC} (\dot{\theta}_m)^2 + \frac{1}{2} m_{BC} \left( \frac{l_{BC}}{2} \right)^2 \dot{\theta}_m^2$$
$$+ \frac{1}{2} T_{BA} \dot{\theta}_m^2 + \frac{1}{2} m_{BA} \left( \frac{l_{AB}}{2} \right)^2 \dot{\theta}_m^2 + \frac{1}{2} my_m^2$$

$$T_{BC} + m_B \left( \frac{l_{BC}}{2} \right)^2 = \frac{1}{12} m_{BC} l_{BC}^2 + \frac{1}{4} m_{BC} l_{BC}^2 = \frac{1}{3} m_{BC} l_{BC}^2$$

$$T_{BA} + m_B \left( \frac{l_{AB}}{2} \right)^2 = \frac{1}{12} m_{AB} l_{AB}^2 + \frac{1}{4} m_{AB} l_{AB}^2 = \frac{1}{3} m_{AB} l_{AB}^2$$
PROBLEM 19.95 (Continued)

\[
m_{BC} = \frac{3}{5} m_{ABC} = 0.45 \text{ kg}
\]

\[
m_{RA} = \frac{2}{5} m_{ABC} = 0.3 \text{ kg}
\]

\[
\frac{1}{3} m_{BC} l_{BC}^2 = \frac{1}{3}(0.45 \text{ kg})(0.3 \text{ m})^2 = 0.0135 \text{ kg} \cdot \text{m}^2
\]

\[
\frac{1}{3} m_{BA} l_{BA}^2 = \frac{1}{3}(0.3)(0.2 \text{ m})^2 = 0.004 \text{ kg} \cdot \text{m}^2
\]

\[
y_m = l_{BC} \dot{\theta}_m
\]

\[
my_m^2 = ml_{BC}^2 \dot{\theta}_m^2 = (1.5 \text{ kg})(0.3 \text{ m})^2 \dot{\theta}_m^2
\]

\[
= 0.135 \dot{\theta}_m^2
\]

\[
T_1 = \frac{1}{2} [0.0135 + 0.004 + 0.135] \dot{\theta}_m^2
\]

\[
= 0.1525 \frac{\dot{\theta}_m^2}{2} \text{ m}
\]

\[
V_1 = \frac{1}{2} k_c (\delta_{ST})_C^2 + \frac{1}{2} k_a (\delta_{ST})_A^2
\]

Position ③

\[
T_2 = 0
\]

\[
V_2 = W l_{BC} \theta_m + W_{BC} l_{BC} \left( 2 \theta_m - \frac{W_{BA}}{l_{BA}} \right) (1 - \cos \theta_m)
\]

\[
+ \frac{1}{2} k_c (l_{BC} \theta_m + (\delta_{ST})_C) \dot{\theta}_m^2 + \frac{1}{2} k_a (l_{AB} \theta_m + (\delta_{ST})_A)^2
\]

when the system is in equilibrium \((\theta = 0)\).

\[
\sum M_B = 0 = W l_{BC} + W_{BC} l_{BC} \frac{1}{2} - k_c (\delta_{ST})_C l_{BC} - k_a (\delta_{ST})_A l_{AB}
\]

\[
1 - \cos \theta_m - 2 \sin^2 \frac{\theta_m}{2} = \frac{\dot{\theta}_m^2}{2}
\]

\[
V_2 = \left[ W l_{BC} + W_{BC} \left( \frac{l_{BC}}{2} \right) \right] \theta_m - \left[ W_{BA} \left( \frac{l_{AB}}{2} \right) \left( \frac{\theta_m^2}{2} \right) \right] - \frac{1}{2} k_c l_{BC} \dot{\theta}_m^2
\]

\[
- k_c (\delta_{ST})_C l_{BC} \dot{\theta}_m + \frac{1}{2} k_c (\delta_{ST})_C^2
\]

\[
+ \frac{1}{2} k_a l_{AB} \dot{\theta}_m^2 - k_a (\delta_{ST})_A l_{AB} \dot{\theta}_m + \frac{1}{2} k_a (\delta_{ST})_A^2
\]
PROBLEM 19.95 (Continued)

Taking Equation (1) into account,

\[
V_2 = - W_B A \left( \frac{l_{AB}}{2} \right) \theta_m^2 + \frac{1}{2} k_C l_{BC} \theta_m^2 + \frac{1}{2} k_C (\delta_{ST})^2_C + \frac{1}{2} k_A l_{AB} \theta_m^2 + \frac{1}{2} k_A (\delta_{ST})^2_A \\
V_2 = \frac{1}{2} \left[ -(0.3)(9.81) \left( \frac{0.2}{2} \right) + (300)(0.3)^2 + 400(0.2)^2 \right] \theta_m^2 + \frac{1}{2} k_C (\delta_{ST})^2_C + \frac{1}{2} k_A (\delta_{ST})^2_A \\
V_2 = \frac{1}{2} [42.7057] \theta_m^2 + \frac{1}{2} k_C (\delta_{ST})^2_C + \frac{1}{2} k_A (\delta_{ST})^2_A
\]

Conservation of energy.

\[
T_1 + V_1 = T_2 + V_2: \quad \frac{1}{2} (0.1525) \theta_m^2 + \frac{1}{2} k_C (\delta_{ST})^2_C + \frac{1}{2} k_A (\delta_{ST})^2_A \\
= 0 + \frac{1}{2} (42.7057) \theta_m^2 + \frac{1}{2} k_C (\delta_{ST})^2_C + \frac{1}{2} k_A (\delta_{ST})^2_A
\]

Simple harmonic motion.

\[
\dot{\theta}_m = \omega_n \theta_m \\
0.1525 \omega_n^2 \theta_m^2 = 42.7057 \theta_m^2 \\
\omega_n^2 = \frac{42.7057}{0.1525} = 280.037 \text{ (rad/s)}^2
\]

Frequency.

\[
f_n = \frac{\omega_n}{2\pi} = \frac{\sqrt{280.037}}{2\pi} \quad f_n = 2.66 \text{ Hz}
\]
PROBLEM 19.96*

Two uniform rods $AB$ and $BC$, each of mass $m$ and length $l$, are pinned together at $A$ and are pin-connected to small rollers at $B$ and $C$. A spring of constant $k$ is attached to the pins at $B$ and $C$, and the system is observed to be in equilibrium when each rod forms an angle $\beta$ with the vertical. Determine the period of small oscillations when Point $A$ is given a small downward deflection and released.

SOLUTION

$x_C = l \sin \beta$

$\delta x_C = l \cos \beta \delta \beta$

$x_G = \frac{l}{2} \cos \beta$

$\delta x_G = -\frac{l}{2} \sin \beta \delta \beta$

$y_G = \frac{l}{2} \sin \beta$

$\delta y_G = \frac{l}{2} \cos \beta \delta \beta$

$\delta y_G = \frac{l}{2} \cos \beta \delta \beta$

**Position 1**

$T_1 = \frac{1}{2} \left[ \frac{1}{2} T (\delta \dot{\beta}_m)^2 + \frac{1}{2} m (\delta \dot{x}_m)^2 + (\delta \dot{y}_m)^2 \right]$  

$= \frac{1}{12} ml^2 + \frac{ml^2}{4} (\sin^2 \beta + \cos^2 \beta) \delta \dot{\beta}_m^2$

$= \frac{1}{3} ml^2 \delta \dot{\beta}_m$

$V_1 = 0$

**Position 2**

$T_2 = 0$

$V_2 = \frac{1}{2} k (2 \delta x_m)^2$

$= \frac{1}{2} (4l^2 \cos^2 \beta)$

$= 2l^2 \cos^2 \beta$

$\delta \dot{\beta}_m = \omega_n \delta \beta_m$
PROBLEM 19.96* (Continued)

\[ T_1 + V_1 = T_2 + V_2 \]

Conservation of energy.

\[ \frac{1}{3} ml^2 \omega_n^2 \delta \beta_n^2 + 0 = 0 + 2kl^2 \cos^2 \beta \delta \beta_n^2 \]

\[ \omega_n^2 = \frac{6k}{m} \cos^2 \beta \]

Period of oscillations.

\[ \tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{6\left(\frac{k}{m}\right)\cos^2 \beta}} \]

\[ \tau_n = \frac{2\pi}{\cos \beta \sqrt{6k}} \]

\[ \Downarrow \]
PROBLEM 19.97*

As a submerged body moves through a fluid, the particles of the fluid flow around the body and thus acquire kinetic energy. In the case of a sphere moving in an ideal fluid, the total kinetic energy acquired by the fluid is \( \frac{1}{2} \rho V v^2 \), where \( \rho \) is the mass density of the fluid, \( V \) is the volume of the sphere, and \( v \) is the velocity of the sphere.

Consider a 500-g hollow spherical shell of radius 80 mm, which is held submerged in a tank of water by a spring of constant 500 N/m. (a) Neglecting fluid friction, determine the period of vibration of the shell when it is displaced vertically and then released. (b) Solve Part a, assuming that the tank is accelerated upward at the constant rate of 8 m/s².

SOLUTION

This is not a damped vibration. However, the kinetic energy of the fluid must be included.

(a) Position ①

\[
T_2 = 0 \\
V_2 = \frac{1}{2} k x_n^2
\]

Position ①

\[
T_1 = T_{\text{spere}} + T_{\text{fluid}} = \frac{1}{2} m_x v_m^2 + \frac{1}{4} \rho V v_m^2
\]

\[
V_1 = 0
\]

Conservation of energy and simple harmonic motion.

\[
T_1 + V_1 = T_2 + V_2: \quad \frac{1}{2} m_x v_m^2 + \frac{1}{4} \rho V v_m^2 + 0 = 0 + \frac{1}{2} k x_n^2
\]

\[
v_m = l_m = x_m \omega_n
\]

\[
\frac{1}{2} \left( m_x + \frac{1}{2} \rho V \right) x_m^2 \omega_n^2 = \frac{1}{2} k x_n^2
\]

\[
\omega_n^2 = \frac{k}{m_x + \frac{1}{2} \rho V}
\]

\[
\omega_n^2 = \frac{500 \text{ N/m}}{(0.5 \text{ kg}) + \left( \frac{1}{2} \rho V \right)}
\]

\[
\frac{1}{2} \rho V = \frac{1}{2} \left( \frac{1000 \text{ kg}}{m^3} \right) \left( \frac{4}{3} \pi (0.08 \text{ m})^3 \right) = 1.0723 \text{ kg}
\]

\[
\omega_n^2 = 318 \text{ s}^{-2}
\]

Period of vibration.

\[
\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{318}}
\]

\[\tau_n = 0.352 \text{ s} \]

(b) Acceleration does not change mass.
PROBLEM 19.98*

A thin plate of length \( l \) rests on a half cylinder of radius \( r \). Derive an expression for the period of small oscillations of the plate.

SOLUTION

\[
(r \sin \theta_m) \sin \theta_m = r \theta_m^2
\]
\[
r(1 - \cos \theta_m) = r \frac{\theta_m^2}{2}
\]

**Position ① (Maximum deflection)**

\[
T_1 = 0
\]
\[
V_1 = W y_m
\]
\[
= mgr \frac{\theta_m^2}{2}
\]

**Position ② (\( \theta = 0 \))**:

\[
T_2 = \frac{1}{2} I \dot{\theta}_m^2
\]
\[
= \frac{1}{2} \left( \frac{1}{12} \right) m l^2 \dot{\theta}_m^2
\]
\[
\dot{\theta}_m = \omega_m \theta_m
\]
\[
T_2 = \frac{1}{2} \left( \frac{1}{12} \right) m l^2 \omega_m^2 \theta_m^2
\]
\[
T_1 + V_1 = T_2 + V_2
\]

\[
0 + \frac{1}{2} m g r \theta_m^2 = \frac{1}{2} \left( \frac{1}{12} \right) m l^2 \omega_m^2 \theta_m^2
\]
\[
\omega_m^2 = \frac{12 g r}{l^2}
\]
\[
\omega_n = \frac{2 \pi}{\sqrt{\frac{l^2}{12 g r}}}
\]
\[
\tau_n = \frac{\pi l}{\sqrt{3 g r}}
\]
PROBLEM 19.99

A 50-kg block is attached to a spring of constant $k = 20 \text{ kN/m}$ and can move without friction in a vertical slot as shown. It is acted upon by a periodic force of magnitude $P = P_m \sin \omega_f t$, where $\omega_f = 18 \text{ rad/s}$. Knowing that the amplitude of the motion is 3 mm, determine the value of $P_m$.

SOLUTION

Equation of motion.

$m\ddot{x} + kx = P_m \sin \omega_f t$

The steady state response is

$x_m = \frac{P_m}{\sqrt{k^2 - \omega_n^2}}

1 - \left(\frac{\omega_f}{\omega_n}\right)^2$

or

$P_m = kx_m \left[1 - \left(\frac{\omega_f}{\omega_n}\right)^2\right]$

Data:

$k = 20 \times 10^3 \text{ N/m}$

$m = 50 \text{ kg}$

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{20 \times 10^3}{50}} = 20 \text{ rad/s}$

$\frac{\omega_f}{\omega_n} = \frac{18 \text{ rad/s}}{20 \text{ rad/s}} = 0.9$

$x_m = 3 \text{ mm} = 3 \times 10^{-3} \text{ m}$

$P_m = (20 \times 10^3)(3 \times 10^{-3})[1 - (0.9)^2]$

$P_m = 11.40 \text{ N}$
PROBLEM 19.100

A 5 kg collar can slide on a frictionless horizontal rod and is attached to a spring of constant 500 N/m. It is acted upon by a periodic force of magnitude $P = P_m \sin \omega_f t$, where $P_m = 15$ N. Determine the amplitude of the motion of the collar if (a) $\omega_f = 5$ rad/s, (b) $\omega_f = 10$ rad/s.

SOLUTION

Eq. (19.33):

$$x_m = \frac{\left(\frac{P_m}{k}\right)}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}$$

$P_m = 15$ N, $k = 500$ N/m

$m = 5$ kg

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{500}{5}} = 10$ rad/s

$\frac{P_m}{k} = \frac{5}{500} = 0.01$ m

Amplitude of vibration.

(a) $\omega_f = 5$ rad/s

$$x_m = \frac{0.03}{1 - (10)^2} = 0.04$$ m

$|x_m| = 0.04$ m (in phase) ◀

(b) $\omega_f = 10$ rad/s

$\omega_f = \omega_n \Rightarrow x_m \rightarrow \infty$

$\omega_f \rightarrow \infty$ ◀
PROBLEM 19.101

A 5-kg collar can slide on a frictionless horizontal rod and is attached to a spring of constant $k$. It is acted upon by a periodic force of magnitude $P = P_m \sin \omega_f t$, where $P_m = 10$ N and $\omega_f = 5$ rad/s. Determine the value of the spring constant $k$ knowing that the motion of the collar has an amplitude of 150 mm and is (a) in phase with the applied force, (b) out of phase with the applied force.

SOLUTION

Eq. (19.33):

$$x_m = \frac{P_m}{k - m\omega_f^2}$$

$$\omega_f^2 = \frac{k}{m}$$

$$x_m = \frac{P_m}{k - m\omega_f^2}$$

$$k = \frac{P_m}{x_m} + m\omega_f^2$$

Data:

$P_m = 10$ N, $m = 5$ kg

$\omega_f = 5$ rad/s

$$k = \frac{P_m}{x_m} + (5)(5)^2$$

$$= \frac{P_m}{x_m} + 125$$

(a) (In phase)

$x_m = 150$ mm = 0.15 m

$$k = \frac{10}{0.15} + 125$$

$k = 191.7$ N/m

(b) (Out of phase)

$x_m = -150$ mm = -0.15 m

$$k = \frac{10}{-0.15} + 125$$

$k = 58.3$ N/m
PROBLEM 19.102

A collar of mass \( m \) which slides on a frictionless horizontal rod is attached to a spring of constant \( k \) and is acted upon by a periodic force of magnitude \( P = P_m \sin \omega_f t \). Determine the range of values of \( \omega_f \) for which the amplitude of the vibration exceeds two times the static deflection caused by a constant force of magnitude \( P_m \).

SOLUTION

Circular natural frequency.

\[ \omega_n = \sqrt{\frac{k}{m}} \]

For forced vibration, the equation of motion is

\[ m\ddot{x} + kx = P_m \sin(\omega_f t + \phi) \]

The amplitude of vibration is

\[ x_m = \frac{P_m}{k} \left( \frac{\omega_n}{\omega_f} \right)^2 \left[ \frac{\delta_{ST}}{1 - \left( \frac{\omega_f}{\omega_n} \right)^2} \right] \]

For \( \omega_f < \omega_n \) and \( x_m = 2 \delta_{ST} \), we have

\[ 2 \delta_{ST} = \frac{\delta_{ST}}{1 - \left( \frac{\omega_f}{\omega_n} \right)^2} \quad \text{or} \quad 1 - \left( \frac{\omega_f}{\omega_n} \right)^2 = \frac{1}{2} \]

\[ \omega_f^2 = \frac{1}{2} \omega_n^2 = \frac{1}{2} \frac{k}{m} \quad \omega_f = \sqrt{\frac{k}{2m}} \]  

(1)

For \( \sqrt{\frac{k}{2m}} < \omega_f \leq \omega_n \), \( |x_m| \) exceeds \( 2\delta_{ST} \)

For \( \omega_f > \omega_n \) and \( x_m = 2\delta_{ST} \), we have

\[ 2\delta_{ST} = \frac{\delta_{ST}}{(\omega_f - \omega_n)^2 - 1} \quad \text{or} \quad \frac{\omega_f^2}{\omega_n^2} - \frac{1}{2} \]

\[ \omega_f^2 = \frac{3}{2} \omega_n^2 = \frac{3}{2} \frac{k}{m} \quad \omega_f = \sqrt{\frac{3k}{2m}} \]  

(2)

For \( \omega_n \leq \omega_f \leq \sqrt{\frac{3k}{2m}} \), \( |x_m| \) exceeds \( 2\delta_{ST} \)

From Eqs. (1) and (2),

\[ \sqrt{\frac{k}{2m}} < \omega_f < \sqrt{\frac{3k}{2m}} \]

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PROBLEM 19.103

An 8-kg uniform disk of radius 200 mm is welded to a vertical shaft with a fixed end at B. The disk rotates through an angle of $3^\circ$ when a static couple of magnitude 50 N·m is applied to it. If the disk is acted upon by a periodic torsional couple of magnitude $T = T_m \sin \omega_f t$, where $T_m = 60$ N·m, determine the range of values of $\omega_f$ for which the amplitude of the vibration is less than the angle of rotation caused by a static couple of magnitude $T_m$.

SOLUTION

Mass moment of inertia: 
$$I = \frac{1}{2} mr^2 = \frac{1}{2} (8)(0.200)^2 = 0.16 \text{ kg} \cdot \text{m}^2$$

Torsional spring constant: 
$$K = \frac{T}{\theta}$$

$T = 50$ N·m
$\theta = 3^\circ = 0.05236$ rad

$$K = \frac{50}{0.05236} = 954.93 \text{ N} \cdot \text{m/rad}$$

Natural circular frequency: 
$$\omega_n = \sqrt{\frac{K}{I}} = \sqrt{\frac{954.93}{0.16}} = 77.254 \text{ rad/s}$$

For forced vibration, 
$$\theta_m = \frac{r_n}{K} = \frac{\theta_{ST}}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}$$

For the amplitude $|\theta_m|$ to be less than $\theta_{ST}$, we must have $\omega_f > \omega_n$.

Then 
$$|\theta_m| = \frac{\theta_{ST}}{\left(\frac{\omega_f}{\omega_n}\right)^2} < \theta_{ST}$$

$$\left(\frac{\omega_f}{\omega_n}\right)^2 > 1$$

$$\left(\frac{\omega_f}{\omega_n}\right)^2 > 2 \quad \omega_f > \sqrt{2}\omega_n = (\sqrt{2})(77.254)$$

$$\omega_f > 109.3 \text{ rad/s} \uparrow$$
PROBLEM 19.104

For the disk of Problem 19.103 determine the range of values of $\omega_f$ for which the amplitude of the vibration will be less than $3.5^\circ$.

SOLUTION

Mass moment of inertia:
\[
T = \frac{1}{2}mr^2 = \frac{1}{2}(8)(0.200)^2 = 0.16 \text{ kg } \cdot \text{m}^2
\]

Torsional spring constant:
\[
K = \frac{T}{\theta} = \frac{50}{0.05236} = 954.93 \text{ N } \cdot \text{m/ rad}
\]

Natural circular frequency:
\[
\omega_n = \sqrt{\frac{K}{T}} = \sqrt{\frac{954.93}{0.16}} = 77.254 \text{ rad/s}
\]

For forced vibration,
\[
\theta_m = \frac{T_m}{K} \left( 1 - \left( \frac{\omega_f}{\omega_n} \right)^2 \right) = \frac{\theta_{ST}}{1 - \left( \frac{\omega_f}{\omega_n} \right)^2}
\]
\[
T_m = 60 \text{ N } \cdot \text{m}
\]
\[
\theta_{ST} = \frac{T_m}{K} = \frac{60}{954.93} = 0.062832 \text{ rad}
\]

For the amplitude $|\theta_m|$ to be less than $\theta_{ST}$, we must have $\omega_f > \omega_n$.

\[
|\theta_m| = \frac{\theta_{ST}}{1 - \left( \frac{\omega_f}{\omega_n} \right)^2} = \frac{0.062832}{1 - \left( \frac{\omega_f}{0.200} \right)^2} < 0.061087
\]

\[
\left( \frac{\omega_f}{\omega_n} \right)^2 - 1 > \frac{0.062832}{0.061087} = 1.02857
\]

\[
\left( \frac{\omega_f}{\omega_n} \right)^2 > 2.02857 \quad \omega_f > \sqrt{2.02857} \cdot \omega_n \quad \omega_f > 110.0 \text{ rad/s}
\]
PROBLEM 19.105

An 8-kg block $A$ slides in a vertical frictionless slot and is connected to a moving support $B$ by means of a spring $AB$ of constant $k = 1.6 \text{ kN/m}$. Knowing that the displacement of the support is $\delta = \delta_m \sin \omega_v t$, where $\delta_m = 150 \text{ mm}$, determine the range of values of $\omega_v$ for which the amplitude of the fluctuating force exerted by the spring on the block is less than $120 \text{ N}$.

SOLUTION

Natural circular frequency:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1.6 \times 10^3}{8}} = 14.1421 \text{ rad/s}$$

Eq. (19.33’):

$$x_m = \frac{\delta_m}{1 - \left(\frac{\omega_v}{\omega_n}\right)^2}$$

Spring force:

$$F_m = -K(x_m - \delta_m) = -k\delta_m \left[ 1 - \frac{1}{1 - \left(\frac{\omega_v}{\omega_n}\right)^2} \right]$$

$$= k\delta_m \frac{\left(\frac{\omega_v}{\omega_n}\right)^2}{1 - \left(\frac{\omega_v}{\omega_n}\right)^2}$$

$$= (1.6 \times 10^3)(0.150) \frac{\left(\frac{\omega_v}{\omega_n}\right)^2}{1 - \left(\frac{\omega_v}{\omega_n}\right)^2} = 240 \frac{\left(\frac{\omega_v}{\omega_n}\right)^2}{1 - \left(\frac{\omega_v}{\omega_n}\right)^2}$$

Limit on spring force:

$$|F_m| < 120 \text{ N}$$

$$240 \left| \frac{\left(\frac{\omega_v}{\omega_n}\right)^2}{1 - \left(\frac{\omega_v}{\omega_n}\right)^2} \right| < 120 \quad \text{or} \quad \left| \frac{\left(\frac{\omega_v}{\omega_n}\right)^2}{1 - \left(\frac{\omega_v}{\omega_n}\right)^2} \right| < \frac{1}{2}$$
PROBLEM 19.105 (Continued)

In phase motion.

\[
\frac{\left( \frac{\omega_f}{\omega_n} \right)^2}{1 - \left( \frac{\omega_f}{\omega_n} \right)^2} < \frac{1}{2}
\]

\[
\left( \frac{\omega_f}{\omega_n} \right)^2 < \frac{1}{2} - \frac{1}{2} \left( \frac{\omega_f}{\omega_n} \right)^2
\]

\[
\frac{3}{2} \left( \frac{\omega_f}{\omega_n} \right)^2 < \frac{1}{2} \quad \frac{\omega_f}{\omega_n} \frac{1}{3}
\]

\[
\omega_f < \frac{1}{\sqrt{3}} \omega_n
\]

\[
\omega_f < 8.16 \text{ rad/s}
\]

Out of phase motion.

\[
\frac{\left( \frac{\omega_f}{\omega_n} \right)^2}{1 - \left( \frac{\omega_f}{\omega_n} \right)^2 - 1} < \frac{1}{2}
\]

\[
\left( \frac{\omega_f}{\omega_n} \right)^2 < 1 \frac{\omega_f}{\omega_n} \frac{1}{2} \left( \frac{\omega_f}{\omega_n} \right)^2 - \frac{1}{2}
\]

\[
\left( \frac{\omega_f}{\omega_n} \right)^2 < -\frac{1}{2}
\]

No solution for \( \omega_f \).
PROBLEM 19.106

Rod $AB$ is rigidly attached to the frame of a motor running at a constant speed. When a collar of mass $m$ is placed on the spring, it is observed to vibrate with an amplitude of 15 mm. When two collars, each of mass $m$, are placed on the spring, the amplitude is observed to be 18 mm. What amplitude of vibration should be expected when three collars, each of mass $m$, are placed on the spring? (Obtain two answers.)

SOLUTION

(a) One collar:

$$(x_m)_1 = 15 \text{ mm} \quad (\omega_n)_1^2 = \frac{k}{m}$$

(b) Two collars:

$$(x_m)_2 = 18 \text{ mm} \quad (\omega_n)_2^2 = \frac{k}{2m} = \frac{1}{2}(\omega_n)_1^2$$

$$\left(\frac{\omega}{\omega_n}\right)_2 = \sqrt{2} \left(\frac{\omega}{\omega_n}\right)_1$$

(c) Three collars:

$$(x_m)_3 = \text{unknown}, \quad (\omega_n)_3^2 = \frac{k}{3m} = \frac{1}{3}(\omega_n)_1^2, \quad \left(\frac{\omega}{\omega_n}\right)_3 = \sqrt{3} \left(\frac{\omega}{\omega_n}\right)_1$$

We also note that the amplitude $\delta_m$ of the displacement of the base remains constant.

Referring to Section 19.7, Figure 19.9, we note that, since $(x_m)_2 > (x_m)_1$ and $\frac{\omega}{\omega_n}_2 > \frac{\omega}{\omega_n}_1$, we must have $\frac{\omega}{\omega_n}_2 < 1$ and $(x_m)_1 > 0$. However, $\frac{\omega}{\omega_n}_2$ may be either $< 1$ or $> 1$, with $(x_m)_2$ being correspondingly either $> 0$ or $< 0$.

1. **Assuming $(x_m)_2 > 0$:**

For one collar,

$$(x_m)_1 = \frac{\delta_m}{1 - \left(\frac{\omega}{\omega_n}\right)_1^2} \hat{j} + 15 \text{ mm} = \frac{\delta_m}{1 - \left(\frac{\omega}{\omega_n}\right)_1^2}$$

(1)

For two collars,

$$(x_m)_2 = \frac{\delta_m}{1 - \left(\frac{\omega}{\omega_n}\right)_2^2} \hat{j} + 18 \text{ mm} = \frac{\delta_m}{1 - 2\left(\frac{\omega}{\omega_n}\right)_1^2}$$

(2)
PROBLEM 19.106 (Continued)

Dividing Eq. (2) by Eq. (1), member by member:

\[
1.2 = \frac{1 - \left( \frac{\omega}{\omega_n} \right)^2}{1 - 2 \left( \frac{\omega}{\omega_n} \right)_1}, \quad \text{we find} \quad \left( \frac{\omega}{\omega_n} \right)_1 = \frac{1}{7}
\]

Substituting into Eq. (1), \( \delta_m = (15 \text{ mm}) \left( 1 - \frac{1}{7} \right) = \frac{90}{7} \text{ mm} \)

For three collars,

\[
(x_m)_3 = \frac{\delta_m}{1 - 3 \left( \frac{\omega}{\omega_n} \right)_1} = \frac{\left( \frac{90}{7} \right) \text{ mm}}{1 - 3 \left( \frac{1}{7} \right)} = \frac{90}{4} \text{ mm}, \quad (x_m)_3 = 22.5 \text{ mm} \quad \blacksquare
\]

2. Assuming \((x_m)_2 < 0\):

For two collars, we have

\[
-18 \text{ mm} = \frac{\delta_m}{1 - 2 \left( \frac{\omega}{\omega_n} \right)_1}
\]

Dividing Eq. (3) by Eq. (1), member by member:

\[
-1.2 = \frac{1 - \left( \frac{\omega}{\omega_n} \right)^2}{1 - 2 \left( \frac{\omega}{\omega_n} \right)_1}
\]

\[
-1.2 + 2.4 \left( \frac{\omega}{\omega_n} \right)_1^2 = 1 - \left( \frac{\omega}{\omega_n} \right)_1^2
\]

\[
\left( \frac{\omega}{\omega_n} \right)_1^2 = \frac{2.2}{3.4} = \frac{1.1}{1.7}
\]

Substitute into Eq. (1), \( \delta_m = (15 \text{ mm}) \left( 1 - \frac{1.1}{1.7} \right) = \frac{9}{1.7} \text{ mm} \)
PROBLEM 19.106 (Continued)

For three collars,

\[
(x_m)_3 = \delta_m \left[1 - \frac{3}{1.1} \right] = \left(\frac{9}{1.7}\right) = 9 \text{ mm} - 1.6 = 7.4 \text{ mm}, \quad (x_m)_3 = -5.63 \text{ mm} \quad (\text{out of phase})
\]

Points corresponding to the two solutions are indicated below:
**PROBLEM 19.107**

A cantilever beam $AB$ supports a block which causes a static deflection of 50 mm at $B$. Assuming that the support at $A$ undergoes a vertical periodic displacement $\delta = \delta_m \sin \omega_f t$, where $\delta_m = 12$ mm, determine the range of values of $\omega_f$ for which the amplitude of the motion of the block will be less than 25 mm. Neglect the weight of the beam and assume that the block does not leave the beam.

**SOLUTION**

For the static condition.  

$$mg = k\delta_{ST}$$

Natural circular frequency.  

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\delta_{ST}}}$$

$$g = 9.81 \text{ m/s}^2, \quad \delta_{ST} = 50 \text{ mm} = 0.05 \text{ m}$$

$$\omega_n = \frac{9.81}{0.05} = 14.007 \text{ rad/s}$$

From Eqs. (19.31 and 19.33’):

$$(x_m)_B = \frac{\delta_m}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}$$

Conditions:  

$$|x_m|_B < 0.025 \text{ m} \quad \delta_m = 0.012 \text{ m}$$

In phase motion.

$$\frac{0.012}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2} < 0.025$$

$$1 - \left(\frac{\omega_f}{\omega_n}\right)^2 > 0.025$$

$$0.52 > \left(\frac{\omega_f}{\omega_n}\right)^2 \quad \omega_f < \sqrt{0.52}\omega_n \quad \omega_f < 10.10 \text{ rad/s} \downarrow$$

Out of phase motion.

$$\frac{0.012}{\left(\frac{\omega_f}{\omega_n}\right)^2 - 1} < 0.025$$

$$\left(\frac{\omega_f}{\omega_n}\right)^2 - 1 > 0.012 \quad \left(\frac{\omega_f}{\omega_n}\right)^2 - 1 > 0.48$$

$$\left(\frac{\omega_f}{\omega_n}\right)^2 > 1.48 \quad \omega_f > \sqrt{1.48}\omega_n \quad \omega_f > 17.04 \text{ rad/s} \downarrow$$
PROBLEM 19.108

A variable-speed motor is rigidly attached to a beam BC. When the speed of the motor is less than 600 rpm or more than 1200 rpm, a small object placed at A is observed to remain in contact with the beam. For speeds between 600 and 1200 rpm the object is observed to “dance” and actually to lose contact with the beam. Determine the speed at which resonance will occur.

SOLUTION

Let \( m \) be the unbalanced mass and \( r \) the eccentricity of the unbalanced mass. The vertical force exerted on the beam due to the rotating unbalanced mass is

\[
P = m\vec{r}\omega_1^2 \sin \omega_1 t = P_m \sin \omega_1 t
\]

Then from Eq. 19.33,

\[
x_m = \frac{P_m}{k} = \frac{m\vec{r}\omega_1^2}{k} \left(1 - \left(\frac{\omega_1}{\omega_n}\right)^2\right)
\]

For simple harmonic motion, the acceleration is

\[
a_m = -\omega_1^2 x_m = \frac{m\vec{r}\omega_1^4}{k} \left(1 - \left(\frac{\omega_1}{\omega_n}\right)^2\right)
\]

When the object loses contact with the beam is \( |a_m| \). Acceleration is greater than \( g \). Let \( \omega_1 = 600 \text{ rpm} = 62.832 \text{ rad/s} \).

\[
|a_m|_1 = \frac{m\vec{r}\omega_1^4}{k} \left(1 - \left(\frac{\omega_1}{\omega_n}\right)^2\right) = \frac{m\vec{r}\omega_1^4}{k} \left(1 - U^2\right)
\]

where

\[
U = \frac{\omega_1}{\omega_n}
\]

Let \( \omega_2 = 1200 \text{ rpm} = 125.664 \text{ rad/s} = 2\omega_1 \)

\[
|a_m|_2 = \frac{m\vec{r}\omega_1^4}{k} \left(\frac{\omega_1}{\omega_n}\right)^2 - 1 = \frac{m\vec{r}\omega_1^4}{k} \left(\frac{\omega_1}{\omega_n}\right)^2 - 1
\]

\[
= \frac{m\vec{r}\omega_1^4}{k} \left(\frac{\omega_1}{\omega_n}\right)^2 - 1 = \frac{m\vec{r}\omega_1^4}{k} \left(\frac{\omega_1}{\omega_n}\right)^2 - 1
\]
PROBLEM 19.108 (Continued)

Dividing Eq. (1) by Eq. (2),

\[
1 = \frac{4U^2 - 1}{16(1 - U^2)} \quad \text{or} \quad 16 - 16U^2 = 4U^2 - 1
\]

\[
20U^2 = 17 \quad U = \frac{17}{20}
\]

\[
\frac{\omega_1}{\omega_n} = \sqrt{\frac{17}{20}} \quad \omega_n = \frac{20}{17} \omega_1 = 1.08465 \omega_1
\]

\[
\omega_n = (1.08465)(600 \text{ rpm}) \quad \omega_n = 651 \text{ rpm}
\]
PROBLEM 19.109

An 8-kg block \( A \) slides in a vertical frictionless slot and is connected to a moving support \( B \) by means of a spring \( AB \) of constant \( k = 120 \text{ N/m} \). Knowing that the acceleration of the support is \( a = a_m \sin \omega_f t \), where \( a_m = 1.5 \text{ m/s}^2 \) and \( \omega_f = 5 \text{ rad/s} \), determine \( (a) \) the maximum displacement of block \( A \), \( (b) \) the amplitude of the fluctuating force exerted by the spring on the block.

SOLUTION

\( (a) \) Support motion.

\[ a = \ddot{\delta} = a_m \sin \omega_f t \]

\[ \delta = -\left( \frac{a_m}{\omega_f^2} \right) \sin \omega_f t \]

\[ \delta_m = -\frac{a_m}{\omega_f^2} = -\frac{1.5 \text{ m/s}^2}{(5 \text{ rad/s})^2} = -0.06 \text{ m} \]

From Equation (19.31 and 19.33'):

\[ x_m = \frac{\delta_m}{\omega_f^2} = \frac{k}{m} = 120 \text{ N/m} \]

\[ x_m = \frac{8 \text{ kg}}{1 - \left( \frac{5}{15} \right)} = 0.09 \text{ m} \]

\[ x_m = 90.0 \text{ mm} \]

\( (b) \) \( x \) is out of phase with \( \delta \) for \( \omega_f = 5 \text{ rad/s} \).

Thus,

\[ F_m = k(x_m + \delta_m) = 120 \text{ N/m}(0.09 \text{ m} + 0.06 \text{ m}) \]

\[ = 18 \text{ N} \]

\[ F_m = 18.00 \text{ N} \]
PROBLEM 19.110

A 20 g ball is connected to a paddle by means of an elastic cord $AB$ of constant $k = 7.5$ N/m. Knowing that the paddle is moved vertically according to the relation $\delta = \delta_m \sin \omega_f t$, where $\delta_m = 200$ mm, determine the maximum allowable circular frequency $\omega_f$ if the cord is not to become slack.

SOLUTION

From Equation (19.31 and 19.33'):

$$x_m = \frac{\delta_m}{\left(1 - \frac{\omega_f^2}{\omega_n^2}\right)}$$

Data:

- $m = 20$ kg $= 0.02$ kg
- $k = 7.5$ N/m
- $\delta_m = 200$ mm $= 0.2$ m

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{7.5}{0.02}} = 19.3649 \text{ rad/s}$$

The cord becomes slack if $x_m - \delta_m$ exceeds $\delta_{ST}$, where

$$\delta_{ST} = \frac{m}{k} \times \frac{0.02 \times 9.81}{7.5} = 0.02616 \text{ m}$$
PROBLEM 19.110 (Continued)

Then

\[
\frac{0.2}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2} - 0.2 < 0.02616
\]

\[
0.2 - 0.2 + 0.2 \left(\frac{\omega_f}{\omega_n}\right)^2 < 0.02616 - 0.02616 \left(\frac{\omega_f}{\omega_n}\right)^2
\]

\[
0.22616 \left(\frac{\omega_f}{\omega_n}\right)^2 < 0.02616
\]

\[
\frac{\omega_f}{\omega_n} < \sqrt{\frac{0.02616}{0.22616}} = 0.3401
\]

Maximum allowable circular frequency.

\[
\omega_f < (0.3401)(19.3649 \text{ rad/s}) \quad \omega_f < 6.59 \text{ rad/s}
\]
PROBLEM 19.111

A simple pendulum of length \( l \) is suspended from a collar \( C \), which is forced to move horizontally according to the relation \( x_C = \delta_m \sin \omega_f t \). Determine the range of values of \( \omega_f \) for which the amplitude of the motion of the bob is less than \( \delta_m \). (Assume that \( \delta_m \) is small compared with the length \( l \) of the pendulum.)

SOLUTION

Geometry.

\[
\begin{align*}
    x &= x_C + l \sin \theta \\
    \sin \theta &= \frac{x - x_C}{l}
\end{align*}
\]

\[
\sum F_y = ma_y: \quad T \cos \theta - mg = 0 \quad T = mg
\]

\[
\sum F_x = ma_x: \quad -T \sin \theta = m \ddot{x}
\]

\[
m \dddot{x} + \frac{mg(x - x_C)}{l} = 0
\]

Using the given motion of \( x_C \),

\[
\dddot{x} + \frac{g}{l} \ddot{x} = \frac{g}{l} \delta_m \sin \omega_f t
\]

Circular natural frequency.

\[
\omega_n = \sqrt{\frac{g}{l}}
\]

\[
\dddot{x} + \omega_n^2 x = \omega_n^2 \delta_m \sin \omega_f t
\]

The steady state response is

\[
x_m = \frac{\delta_m}{1 - \left( \frac{\omega_f}{\omega_n} \right)^2}
\]

\[
x_m^2 = \frac{\delta_m^2}{1 - \left( \frac{\omega_f}{\omega_n} \right)^2} \leq \delta_m^2
\]

Consider

\[
x_m^2 = \delta_m^2.
\]
PROBLEM 19.111 (Continued)

Then

\[
1 - \left( \frac{\omega_f}{\omega_n} \right)^2 = 1 - 2 \left( \frac{\omega_f}{\omega_n} \right)^2 + \left( \frac{\omega_f}{\omega_n} \right)^4 = 1
\]

\[
\left( \frac{\omega_f}{\omega_n} \right)^2 = 0 \quad \text{and} \quad \left( \frac{\omega_f}{\omega_n} \right)^2 = 2
\]

\[
\left( \frac{\omega_f}{\omega_n} \right)^2 = 0 \quad \text{and} \quad \frac{\omega_f}{\omega_n} = \sqrt{2}
\]

For

\[
0 < \frac{\omega_f}{\omega_n} < \sqrt{2}, \quad |x_m| > \delta_m
\]

For

\[
\frac{\omega_f}{\omega_n} > \sqrt{2}, \quad |x_m| < \delta_n
\]

Then

\[
\omega_f > \sqrt{2} \omega_n = \sqrt{\frac{2g}{l}} \quad \omega_f > \sqrt{\frac{2g}{l}} \nabla
\]
PROBLEM 19.112

The 1.2-kg bob of a simple pendulum of length \( l = 600 \text{ mm} \) is suspended from a 1.4-kg collar \( C \). The collar is forced to move according to the relation \( x_C = \delta_m \sin \omega_f t \), with an amplitude \( \delta_m = 10 \text{ mm} \) and a frequency \( \omega_f = 0.5 \text{ Hz} \). Determine (a) the amplitude of the motion of the bob, (b) the force that must be applied to collar \( C \) to maintain the motion.

SOLUTION

(a)

\[
\sum F_x = ma_x \\
- T \sin \theta = m \ddot{x} \\
\sum F_y = T \cos \theta - mg = 0
\]

For small angles \( \cos \theta = 1 \). Acceleration in the \( y \) direction is second order and is neglected.

\[
T = mg \\
m \ddot{x} = -mg \sin \theta \\
\sin \theta = \frac{x - x_c}{l} \\
m \ddot{x} + \frac{mg}{l} x = \frac{g}{l} x_c = \frac{mg}{l} \delta m \sin \omega_f t \\
\omega_n^2 = \frac{g}{l} \\
\ddot{x} + \omega_n^2 x = \omega_n^2 \delta m \sin \omega_f t
\]

From Equation (19.33’):

\[
x_m = \frac{\delta m}{1 - \frac{\omega_n^2}{\omega_f^2}}
\]

So

\[
\omega_f^2 = (2\pi \omega_f)^2 = 4\pi^2 (0.5)^2 = \pi^2 \text{ s}^{-2} \\
\omega_n^2 = \frac{g}{l} = \frac{9.81 \text{ m/s}^2}{0.600 \text{ m}} = 16.35 \text{ s}^{-2} \\
x_m = \frac{10 \times 10^{-3} \text{ m}}{1 - \frac{\pi^2}{16.35}} = 0.02523 \text{ m} \\
x_m = 25.2 \text{ mm} \]
PROBLEM 19.112 (Continued)

(b)

\[ a_c = \ddot{x}_c = -\delta m \omega_f^2 \sin \omega_f t \]

\[ \sum F_x = m_c \ddot{x}_c \]

\[ F - T \sin \theta = m_c \ddot{x}_c \]

From Part (a):

\[ T = mg, \quad \sin \theta = \frac{x - x_c}{l} \]

Thus,

\[ F = -mg \left[ \frac{x - x_c}{l} \right] + m_c \ddot{x}_c \]

\[ = -m \omega_n^2 x + m \omega_n^2 x_c + m_c \ddot{x}_c \]

\[ = -m \omega_n^2 x_m \sin \omega_f t + m \omega_n^2 \delta_m \sin \omega_f t - m_c \omega_f^2 \delta_m \sin \omega_f t \]

\[ = -[(1.2)(16.35)(0.02523) + (1.2)(16.35)(10 \times 10^{-3}) - (1.4)\pi^2 (10 \times 10^{-3})] \sin \pi t \]

\[ = -0.437 \sin \pi t \]

\[ F = -0.437 \sin \pi t \text{ (N)} \]
PROBLEM 19.113

A motor of mass $M$ is supported by springs with an equivalent spring constant $k$. The unbalance of its rotor is equivalent to a mass $m$ located at a distance $r$ from the axis of rotation. Show that when the angular velocity of the motor is $\omega_f$, the amplitude $x_m$ of the motion of the motor is

$$x_m = \frac{r \left( \frac{m}{M} \right) \left( \frac{\omega_f}{\omega_n} \right)^2}{1 - \left( \frac{\omega_f}{\omega_n} \right)^2}$$

where $\omega_n = \sqrt{\frac{k}{M}}$

SOLUTION

$$\sum F = ma \quad P_m \sin \omega_f t - kx = M\ddot{x}$$

$$M\ddot{x} + kx = P_m \sin \omega_f t$$

$$\ddot{x} + \frac{k}{m} x = \frac{P_m}{m} \sin \omega_f t$$

$$\omega_n^2 = \frac{k}{M}$$

From Equation (19.33):

$$x_m = \frac{\frac{P_m}{k}}{1 - \left( \frac{\omega_f}{\omega_n} \right)^2}$$

But

$$\frac{P_m}{k} = \frac{m \omega_f^2}{k} \quad k = M \omega_n^2$$

$$\frac{P_m}{k} = r \left( \frac{m}{M} \right) \left( \frac{\omega_f}{\omega_n} \right)^2$$

Thus,

$$x_m = \frac{r \left( \frac{m}{M} \right) \left( \frac{\omega_f}{\omega_n} \right)^2}{1 - \left( \frac{\omega_f}{\omega_n} \right)^2} \quad \text{Q.E.D.}$$
PROBLEM 19.114

As the rotational speed of a spring-supported 100-kg motor is increased, the amplitude of the vibration due to the unbalance of its 15-kg rotor first increases and then decreases. It is observed that as very high speeds are reached, the amplitude of the vibration approaches 3.3 mm. Determine the distance between the mass center of the rotor and its axis of rotation. (Hint: Use the formula derived in Problem 19.113.)

SOLUTION

Use the equation derived in Problem 19.113 (above).

\[
x_m = \frac{r \left( \frac{m}{M} \right) \left( \frac{\omega_f}{\omega_n} \right)^2}{1 - \left( \frac{\omega_f}{\omega_n} \right)^2} = \frac{r \left( \frac{m}{M} \right)}{1 - \left( \frac{\omega_f}{\omega_n} \right)^2 - 1}
\]

For very high speeds, \( \frac{1}{\left( \frac{\omega_f}{\omega_n} \right)^2} \rightarrow 0 \) and \( x_m \rightarrow \frac{rm}{M} \),

thus,

\[
3.3 \text{ mm} = r \left( \frac{15}{100} \right)
\]

\[r = 22 \text{ mm} \]
PROBLEM 19.115

A motor of mass 200 kg is supported by springs having a total constant of 240 kN/m. The unbalance of the rotor is equivalent to a 30-g mass located 200 mm from the axis of rotation. Determine the range of allowable values of the motor speed if the amplitude of the vibration is not to exceed 1.5 mm.

SOLUTION

Let $M =$ mass of motor, $m =$ unbalance mass, $r =$ eccentricity

$M = 200 \text{ kg}$

$m = 30 \text{ g} = 0.03 \text{ kg}$

$r = 200 \text{ mm} = 0.2 \text{ m}$ $K = 240 \text{ kN/m} = 240 \times 10^3 \text{ N/m}$

Natural circular frequency:

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{240 \times 10^3 \text{ N/m}}{200}} = 34.641 \text{ rad/s}$$

$$\frac{rm}{M} = \frac{(0.2)(0.03)}{200} = 0.3 \times 10^{-5} = 0.03 \text{ mm}$$

From the derivation given in Problem 19.113,

$$x_m = \left(\frac{rm}{M}\right)\left(\frac{\omega_f}{\omega_n}\right)^2 = \frac{0.03}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2} \text{ mm}$$

In phase motion with $|x_m| < 1.5 \text{ mm}$

$$\frac{0.03}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2} < 1.5$$

$$0.03\left(\frac{\omega_f}{\omega_n}\right)^2 < 1.5 - 1.5\left(\frac{\omega_f}{\omega_n}\right)^2$$

$$1.53\left(\frac{\omega_f}{\omega_n}\right)^2 < 1.5$$

$$\frac{\omega_f}{\omega_n} < \sqrt{\frac{1.5}{1.53}} = 0.99015$$

$$\omega_f < (0.99015)(34.641) = 34.30 \text{ rad/s}$$

$\omega_f < 328 \text{ rpm}$
PROBLEM 19.115 (Continued)

Out of phase motion with $|x_m| = 1.5$ mm

\[
0.03 \left( \frac{\omega_f}{\omega_n} \right)^2 < 1.5
\]

\[
\frac{\omega_f}{\omega_n} < 1.5 \left( \frac{\omega_f}{\omega_n} \right) - 1.5
\]

\[
1.5 < 1.47 \left( \frac{\omega_f}{\omega_n} \right)^2
\]

\[
\frac{\omega_f}{\omega_n} > \sqrt{\frac{1.5}{1.47}} = 1.01015
\]

\[
\omega_f > (1.01015)(34.641) = 34.992 \text{ rad/s} \quad \omega_f > 334 \text{ rpm}
\]
**PROBLEM 19.116**

As the rotational speed of a spring-supported motor is slowly increased from 300 to 500 rpm, the amplitude of the vibration due to the unbalance of its rotor is observed to increase continuously from 1.5 to 6 mm. Determine the speed at which resonance will occur.

**SOLUTION**

Let $m'$ be the mass of rotor of the motor and $m$ the total mass of the motor. Let $e$ be the distance from the rotor axis to the mass center of the rotor. The magnitude of centrifugal force due to unbalance of the rotor is

$$P_m = m' e \omega_f^2$$

and in rotation, the unbalanced force is

$$P_m \sin \omega_f t = m' e \omega_f^2 \sin \omega_f t$$

The equation of motion of the spring mounted motor is

$$m\ddot{x} + kx = m' e \omega_f^2 \sin \omega_f t = P_m \sin \omega_f t$$

The steady state response is

$$x_m = \frac{P_m}{k} \frac{1}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}$$

$$x_m = \frac{m' e \omega_f^2}{k} \frac{1}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2} = \frac{C \omega_f^2}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}$$

**Case 1.**

$\omega_f = \omega_1 = 300 \text{ rpm} = \frac{2\pi(300)}{60} = 10\pi \text{ rad/s}$

$|x_m| = x_1 = 1.5 \text{ mm}$

**Case 2.**

$\omega_f = \omega_2 = 500 \text{ rpm} = \frac{2\pi(500)}{60} = \frac{50\pi}{3} \text{ rad/s}$

$|x_m| = x_2 = 6 \text{ mm}$

$$\frac{x_2}{x_1} = \frac{6}{1.5} = 4 = \frac{\omega_2^2}{\omega_1^2} \left[ 1 + \left(\frac{\omega_f}{\omega_n}\right)^2 \right]$$

$$4\omega_1^2 \left[ 1 - \left(\frac{\omega_2}{\omega_n}\right)^2 \right] = \omega_2^2 \left[ 1 - 1 \left(\frac{\omega_1}{\omega_n}\right)^2 \right]$$
PROBLEM 19.116 (Continued)

Multiplying by $\omega_n^2$ and transposing terms,

$$(4\omega_1^2 - \omega_2^2)\omega_n^2 - 3\omega_1^2 \omega_2^2 = 0$$

$$\omega_n^2 = \frac{3\omega_1^2 \omega_2^2}{4\omega_1^2 - \omega_2^2} = \frac{(3)(986.96)(2741.56)}{3947.84 - 2741.56} = 6729.3 \text{ (rad/s)}^2$$

$$\omega_n = 82.032 \text{ rad/s} \hspace{1cm} \omega_n = 783 \text{ rpm}$$
PROBLEM 19.117

A 100 kg motor is bolted to a light horizontal beam. The unbalance of its rotor is equivalent to a mass of 60 g located 100 mm from the axis of rotation. Knowing that resonance occurs at a motor speed of 400 rpm, determine the amplitude of the steady-state vibration at (a) 800 rpm, (b) 200 rpm, (c) 425 rpm.

SOLUTION

From Problem 19.113:

\[
x_m = \frac{r \left( \frac{m}{M} \right) \left( \frac{\omega_f}{\omega_n} \right)^2}{1 - \left( \frac{\omega_f}{\omega_n} \right)^2}
\]

Resonance at 400 rpm means that \( \omega_n = 400 \text{ rpm} \).

\[
r \left( \frac{m}{M} \right) = (100 \text{ mm}) \left( \frac{60 \times 10^{-3}}{100} \right) = 0.06 \text{ mm}
\]

(a)

\[
\left( \frac{\omega_f}{\omega_n} \right)^2 = \left( \frac{800}{400} \right)^2 = 4
\]

\[
x_m = \frac{0.06(4)}{1 - 4}
\]

\( x_m = 0.06(4) \)

\( x_m = -0.080 \text{ mm} \)

(b)

\[
\left( \frac{\omega_f}{\omega_n} \right)^2 = \left( \frac{200}{400} \right)^2 = \frac{1}{4}
\]

\[
x_m = \frac{0.06 \left( \frac{1}{4} \right)}{1 - \frac{1}{4}}
\]

\( x_m = 0.06 \left( \frac{1}{4} \right) \)

\( x_m = 0.020 \text{ mm} \)

(c)

\[
\left( \frac{\omega_f}{\omega_n} \right)^2 = \left( \frac{425}{400} \right)^2 = 1.1289
\]

\[
x_m = \frac{0.06(1.1289)}{1 - 1.1289}
\]

\( x_m = 0.06(1.1289) \)

\( x_m = -0.525 \text{ mm} \)
PROBLEM 19.118

A 180-kg motor is bolted to a light horizontal beam. The unbalance of its rotor is equivalent to a 28-g mass located 150 mm from the axis of rotation, and the static deflection of the beam due to the weight of the motor is 12 mm. The amplitude of the vibration due to the unbalance can be decreased by adding a plate to the base of the motor. If the amplitude of vibration is to be less than 60 $\mu$m for motor speeds above 300 rpm, determine the required mass of the plate.

SOLUTION

Before the plate is added, \( M_1 = 180 \text{ kg}, \quad m = 28 \times 10^{-3} \text{ kg} \)
\( r = 150 \text{ mm} = 0.150 \text{ m} \)

Equivalent spring constant:
\[
k = \frac{W_1}{\delta_{ST}} = \frac{M_1 g}{\delta_{ST}}
\]
\[
k = \frac{(180)(9.81)}{12 \times 10^{-3}} = 147.15 \times 10^3 \text{ N/m}
\]

Let \( M_2 \) be the mass of motor plus the plate.

Natural circular frequency.
\[
\omega_n = \sqrt{\frac{k}{M_2}}
\]

Forcing frequency:
\[
\omega_f = 300 \text{ rpm} = 31.416 \text{ rad/s}
\]
\[
\left( \frac{\omega_f}{\omega_n} \right)^2 = \frac{\omega_f^2 M_2}{k} = \frac{(31.416)^2 M_2}{147.15 \times 10^3} = 0.006707 M_2
\]

From the derivation in Problem 19.113,
\[
x_m = \left( \frac{\omega_f}{\omega_n} \right)^2
\]

For out of phase motion with \( x_m = -60 \times 10^{-6} \text{ m} \),
\[
-60 \times 10^{-6} = \frac{(0.150)(28 \times 10^{-3})}{M_2}(0.006707 M_2)
\]
\[
-60 \times 10^{-6} + (60 \times 10^{-6})(0.006707) M_2 = 28.170 \times 10^{-6}
\]
\[
402.49 \times 10^{-9} M_2 = 88.170 \times 10^{-6}
\]
\[
M_2 = 219.10 \text{ kg}
\]

Added mass:
\[
\Delta M = M_2 - M_1 = 219.10 - 180 \quad \Delta M = 39.1 \text{ kg}
\]
PROBLEM 19.119

The unbalance of the rotor of a 200-kg motor is equivalent to a 100-g mass located 150 mm from the axis of rotation. In order to limit to 1 N the amplitude of the fluctuating force exerted on the foundation when the motor is run at speeds of 100 rpm and above, a pad is to be placed between the motor and the foundation. Determine (a) the maximum allowable spring constant \( k \) of the pad, (b) the corresponding amplitude of the fluctuating force exerted on the foundation when the motor is run at 200 rpm.

SOLUTION

Mass of motor: \( M = 200 \) kg

Unbalance mass: \( m = 100 \) g = 0.1 kg

Eccentricity: \( r = 150 \) mm = 0.15 m

Equation of motion:
\[
Mx + kx = P_m \sin \omega f t = m r \omega f^2 \sin \omega f t
\]
\[
(-M \omega f^2 + k)x_m = m r \omega f^2
\]
\[
x_m = \frac{m r \omega f^2}{k - M \omega f^2}
\]

Transmitted force: \( F_m = kx_m = \frac{k m r \omega f^2}{k - M \omega f^2} \)

For out of phase motion,
\[
|F_m| = \frac{k m r \omega f^2}{M \omega f^2 - k}
\]  \hspace{1cm} (1)

(a) Required value of \( k \).

Solve Eq. (1) for \( k \).
\[
|F_m| (M \omega f^2 - k) = km r \omega f^2
\]
\[
k (m r \omega f^2 + |F_m|) = |F_m| M \omega f^2
\]
\[
k = \frac{|F_m| M \omega f^2}{m r \omega f^2 + |F_m|}
\]

Data: \( |F_m| = 1 \) N \hspace{1cm} \( \omega f = 100 \) rpm = 10.472 rad/s

\[
k = \frac{(1)(200)(10.472)^2}{(0.1)(0.15)(10.472)^2 + 1} = 8292.3 \) N/m \hspace{1cm} k = 8.292 kN/m
\]
PROBLEM 19.119 (Continued)

(b) Force amplitude at 200 rpm.

\( \omega_f = 20.944 \text{ rad/s} \)

From Eq. (1),

\[
|F_m| = \frac{(8292.3)(0.1)(0.15)(20.944)^2}{(200)(20.944)^2 - 8292.3} \\
|F_m| = 0.687 \text{ N} \]
PROBLEM 19.120

A 180-kg motor is supported by springs of total constant 150 kN/m. The unbalance of the rotor is equivalent to a 28-g mass located 150 mm from the axis of rotation. Determine the range of speeds of the motor for which the amplitude of the fluctuating force exerted on the foundation is less than 20 N.

SOLUTION

Equation derived in Problem 19.113:

\[ x_m = \frac{\left( \frac{r m}{M} \right) \left( \frac{\omega_f}{\omega_n} \right)^2}{1 - \left( \frac{\omega_f}{\omega_n} \right)^2} \]

Force transmitted:

\[ F_m = k x_m = \frac{\left( \frac{k r m}{M} \right) \left( \frac{\omega_f}{\omega_n} \right)^2}{1 - \left( \frac{\omega_f}{\omega_n} \right)^2} \]

Data:

\[ k = 150 \text{ kN/m} = 150 \times 10^3 \text{ N/m} \]
\[ r = 150 \text{ mm} = 0.150 \text{ m} \]
\[ m = 28 \text{ g} = 28 \times 10^{-3} \text{ kg} \]
\[ M = 180 \text{ kg} \]
\[ \frac{k r m}{M} = \frac{(150 \times 10^3)(0.150)(28 \times 10^{-3})}{180} = 3.5 \text{ N} \]
\[ \omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{150 \times 10^3}{180}} = 28.868 \text{ rad/s} \]

In phase motion with \(|F_m| < 20 \text{ N}.

\[ \frac{3.5 \left( \frac{\omega_f}{\omega_n} \right)^2}{1 - \left( \frac{\omega_f}{\omega_n} \right)^2} < 20 \]
\[ 3.5 \left( \frac{\omega_f}{\omega_n} \right)^2 < 20 - 20 \left( \frac{\omega_f}{\omega_n} \right)^2 \]
\[ 23.5 \left( \frac{\omega_f}{\omega_n} \right)^2 < 20 \]
\[ \frac{\omega_f}{\omega_n} < \sqrt{\frac{20}{23.5}} \]
\[ \omega_f < \sqrt{\frac{20}{23.5}} (28.868) = 26.63 \text{ rad/s} \quad \omega_f < 254 \text{ rpm} \]
PROBLEM 19.120 (Continued)

Out of phase motion with \( |F_m| < 20 \) N.

\[
\begin{align*}
3.5 \left( \frac{\omega_f}{\omega_n} \right)^2 &< 20 \\
\left( \frac{\omega_f}{\omega_n} \right)^2 - 1 &< 20 \\
3.5 \left( \frac{\omega_f}{\omega_n} \right)^2 &< 20 \left( \frac{\omega_f}{\omega_n} \right)^2 - 20 \\
&< 16.5 \left( \frac{\omega_f}{\omega_n} \right)^2
\end{align*}
\]

\[
\frac{\omega_f}{\omega_n} > \sqrt{\frac{20}{16.5}} \quad \omega_f > \sqrt{\frac{20}{16.5}} (28.868) = 31.78 \text{ rad/s} \quad \omega_f > 304 \text{ rpm}
\]
PROBLEM 19.121

A vibrometer used to measure the amplitude of vibrations consists essentially of a box containing a mass-spring system with a known natural frequency of 120 Hz. The box is rigidly attached to a surface, which is moving according to the equation \( y = \delta_m \sin \omega_f t \). If the amplitude \( z_m \) of the motion of the mass relative to the box is used as a measure of the amplitude \( \delta_m \) of the vibration of the surface, determine (a) the percent error when the frequency of the vibration is 600 Hz, (b) the frequency at which the error is zero.

SOLUTION

\[
x = \left( \frac{\delta_m}{1 - \omega_f^2/\omega_n^2} \right) \sin \omega_f t
\]
\[
y = \delta_m \sin \omega_f t
\]
\[
z = \text{relative motion}
\]
\[
z = x - y = \left[ \frac{\delta_m}{1 - \omega_f^2/\omega_n^2} - \delta_m \right] \sin \omega_f t
\]
\[
z_m = \delta_m \left[ \frac{1}{1 - \omega_f^2/\omega_n^2} - 1 \right] = \frac{\delta_m \omega_f^2}{\omega_n^2 - \omega_f^2}
\]

(a)
\[
\frac{z_m}{\delta_m} \frac{\omega_f^2}{\omega_n^2} = \left( \frac{600}{120} \right)^2 = \frac{25}{24} = 1.0417
\]

Error = 4.17%

(b)
\[
\frac{z_m}{\delta_m} = 1 = \frac{\omega_f^2}{\omega_n^2}
\]
\[
1 = 2 \frac{\omega_f^2}{\omega_n^2}
\]
\[
f_f = \frac{\sqrt{2}}{2} f_n = \frac{\sqrt{2}}{2} (120) = 84.853 \text{ Hz}
\]
\[
f_n = 84.9 \text{ Hz}
\]
PROBLEM 19.122

A certain accelerometer consists essentially of a box containing a mass-spring system with a known natural frequency of 2200 Hz. The box is rigidly attached to a surface, which is moving according to the equation $y = \delta_m \sin \omega_f t$. If the amplitude $z_m$ of the motion of the mass relative to the box times a scale factor $\omega_m$ is used as a measure of the maximum acceleration $a_m = \delta_m \omega_f^2$ of the vibrating surface, determine the percent error when the frequency of the vibration is 600 Hz.

SOLUTION

$$x = \left( \frac{\delta_m}{1 - \frac{\omega_f^2}{\omega_m^2}} \right) \sin \omega_f t$$

$$y = \delta_m \sin \omega_f t$$

$$z = \text{relative motion}$$

$$z = x - y = \left[ \frac{\delta_m}{1 - \frac{\omega_f^2}{\omega_m^2}} - \delta_m \right] \sin \omega_f t$$

$$z_m = \delta_m \left[ \frac{1}{1 - \frac{\omega_f^2}{\omega_m^2}} - 1 \right] = \frac{\delta_m \omega_m^2}{\omega_m^2 - \omega_f^2}$$

The actual acceleration is $a_m = -\omega_f^2 \delta_m$.

The measurement is proportional to $z_m \omega_m^2$.

Then

$$\frac{z_m \omega_m^2}{a_m} = \frac{\delta_m \left( \frac{\omega_m}{\omega_f} \right)^2}{\delta_m} = \frac{1}{1 - \left( \frac{\omega_f}{\omega_m} \right)^2}$$

$$= \frac{1}{1 - \left( \frac{600}{2200} \right)^2}$$

$$= 1.0804$$

Error = 8.04%
PROBLEM 19.123

Figures (1) and (2) show how springs can be used to support a block in two different situations. In Figure (1), they help decrease the amplitude of the fluctuating force transmitted by the block to the foundation. In Figure (2), they help decrease the amplitude of the fluctuating displacement transmitted by the foundation to the block. The ratio of the transmitted force to the impressed force or the ratio of the transmitted displacement to the impressed displacement is called the \textit{transmissibility}. Derive an equation for the transmissibility for each situation. Give your answer in terms of the ratio \( \omega_f/\omega_n \) of the frequency \( \omega_f \) of the impressed force or impressed displacement to the natural frequency \( \omega_n \) of the spring-mass system. Show that in order to cause any reduction in transmissibility, the ratio \( \omega_f/\omega_n \) must be greater than \( \sqrt{2} \).

SOLUTION

(1) From Equation (19.33):

\[
x_m = \frac{P_m}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}
\]

Force transmitted:

\[
(P_T)_m = kx_m = k\left[\frac{P_n}{k} \frac{1}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}\right]
\]

Thus,

\[
\text{Transmissibility} = \frac{(P_T)_m}{P_m} = \frac{1}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}
\]

(2) From Equation (19.33’):

Displacement transmitted:

\[
x_m = \frac{\delta_m}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}
\]

\[
\text{Transmissibility} = \frac{x_m}{\delta_m} = \frac{1}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}
\]

For \( \frac{(P_T)_m}{P_m} \) or \( \frac{x_m}{\delta_m} \) to be less than 1,

\[
\frac{1}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2} < 1
\]

\[
1 < \left|\frac{\omega_f}{\omega_n}\right|^2
\]

\[
\frac{\omega_f}{\omega_n} > \sqrt{2}
\]

Q.E.D.
PROBLEM 19.124

A 30-kg disk is attached with an eccentricity $e = 0.15$ mm to the midpoint of a vertical shaft $AB$, which revolves at a constant angular velocity $\omega_f$. Knowing that the spring constant $k$ for horizontal movement of the disk is 650 kN/m, determine (a) the angular velocity $\omega_f$ at which resonance will occur, (b) the deflection $r$ of the shaft when $\omega_f = 1200$ rpm.

SOLUTION

$G$ describes a circle about the axis $AB$ of radius $r + e$.

Thus,

$$a_n = (r + e)\omega_f^2$$

Deflection of the shaft is

$$F = kr$$

Thus,

$$F = ma_n$$

$$kr = m(r + e)\omega_f^2$$

$$\omega_n^2 = \frac{k}{m} \quad m = \frac{k}{\omega_n^2}$$

$$\frac{kr}{\omega_n^2} = \frac{k}{\omega_f^2} (r + e)\omega_f^2$$

$$r = \frac{e}{\omega_n^2} \frac{\omega_f^2}{\omega_f^2}$$

$$1 - \frac{\omega_n^2}{\omega_f^2}$$
**PROBLEM 19.124 (Continued)**

(a) Resonance occurs when

$$\omega_f = \omega_n, \text{ i.e., } r \to \infty$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$= \sqrt{\frac{(650 \times 1000)}{30}}$$

$$= 147.196 \text{ rad/s}$$

$$= 1405.6 \text{ rpm}$$

$$\omega_n = \omega_f = 1406 \text{ rpm} \uparrow$$

(b)

$$r = \frac{(0.15 \text{ mm})(\frac{1200}{1405.6})^2}{1 - \left(\frac{1200}{1405.6}\right)^2}$$

$$= 0.02964 \text{ mm}$$

$$r = 0.403 \text{ mm} \uparrow$$
PROBLEM 19.125

A small trailer and its load have a total mass of 250 kg. The trailer is supported by two springs, each of constant 10 kN/m, and is pulled over a road, the surface of which can be approximated by a sine curve with an amplitude of 40 mm and a wavelength of 5 m (i.e., the distance between successive crests is 5 m and the vertical distance from crest to trough is 80 mm). Determine (a) the speed at which resonance will occur, (b) the amplitude of the vibration of the trailer at a speed of 50 km/h.

SOLUTION

Total spring constant

\[ k = 2(10 \times 10^3) \text{ N/m} = 20 \times 10^3 \text{ N/m} \]

(a) Resonance:

\[ \omega_n = \frac{k}{m} = \frac{20 \times 10^3}{250} = 80 \text{ s}^{-2} \]

\[ \lambda = 5 \text{ m} \]

\[ \delta_m = 40 \text{ mm} = 40 \times 10^{-3} \text{ m} \]

\[ y = \delta_m \sin \frac{x}{\lambda} \]

\[ x = vt \]

\[ y = \delta_m \sin \left( \frac{v}{\lambda} \right)t \quad \text{and} \quad \omega_f = \frac{2\pi}{\tau}, \]

so

\[ y = 40 \times 10^{-3} \sin \omega_f t \]

From Equation (19.33'):

\[ x_m = \frac{\delta_m}{\left(1 - \frac{\omega_f^2}{\omega_n^2}\right)} \]

Resonance:

\[ \omega_f = \frac{2\pi v}{\lambda} = \omega_n = \sqrt{80} \text{ s}^{-1}, \]

\[ v = 7.1176 \text{ m/s} \]

\[ v = 25.6 \text{ km/h} \]

(b) Amplitude at

\[ \omega_f = \frac{2\pi(13.8889)}{5} = 17.4533 \text{ rad/s} \]

\[ \omega_f^2 = 304.60 \text{ s}^{-2} \]

\[ x_m = \frac{40 \times 10^{-3}}{1 - \frac{304.62}{80}} = -14.246 \times 10^{-3} \text{ m} \]

\[ x_m = -14.25 \text{ mm} \]
PROBLEM 19.126

Block $A$ can move without friction in the slot as shown and is acted upon by a vertical periodic force of magnitude $P = P_m \sin \omega_f t$, where $\omega_f = 2$ rad/s and $P_m = 20$ N. A spring of constant $k$ is attached to the bottom of block $A$ and to a 22-kg block $B$. Determine (a) the value of the constant $k$ which will prevent a steady-state vibration of block $A$, (b) the corresponding amplitude of the vibration of block $B$.

SOLUTION

In steady state vibration, block $A$ does not move and therefore, remains in its original equilibrium position.

Block $A$:

$$\Sigma F = 0$$

$$kx = -P_m \sin \omega_f t$$

Block $B$:

$$\Sigma F = m_B \ddot{x}$$

$$m_B \ddot{x} + kx = 0$$

$$x = x_m \sin \omega_n t$$

$$\omega_n^2 = k/m_B$$

From Eq. (1):

$$k x_m \sin \omega_n t = -P_m \sin \omega_f t$$

$$\omega_n = \omega_f = 2 \text{ rad/s}$$

$$k x_m = -P_m$$

$$k = \frac{m_B \omega_n^2}{P_m}$$

(a) Required spring constant.

$$k = (22)(2)^2$$

$$k = 88.0 \text{ N/m}$$

(b) Corresponding amplitude of vibration of $B$.

$$k x_m = -P_m$$

$$x_m = \frac{-P_m}{k}$$

$$x_m = -\frac{20 \text{ N}}{88 \text{ N/m}}$$

$$x_m = -0.227 \text{ N/m}$$

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PROBLEM 19.127

Show that in the case of heavy damping \((c > c_c)\), a body never passes through its position of equilibrium \(O\) (a) if it is released with no initial velocity from an arbitrary position or (b) if it is started from \(O\) with an arbitrary initial velocity.

SOLUTION

Since \(c > c_c\), we use Equation (19.42), where

\[
\lambda_1 < 0, \quad \lambda_2 < 0
\]

\[
x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}
\]

(1)

\[
v = \frac{dx}{dt} = c_1 \lambda_1 e^{\lambda_1 t} + c_2 \lambda_2 e^{\lambda_2 t}
\]

(2)

(a) \(t = 0, \quad x = x_0, \quad v = 0:\)

From Eqs. (1) and (2):

\[
x_0 = c_1 + c_2
\]

\[
0 = c_1 \lambda_1 + c_2 \lambda_2
\]

Solving for \(c_1\) and \(c_2\),

\[
c_1 = \frac{-\lambda_2}{\lambda_2 - \lambda_1} x_0
\]

\[
c_2 = \frac{-\lambda_1}{\lambda_2 - \lambda_1} x_0
\]

Substituting for \(c_1\) and \(c_2\) in Eq. (1),

\[
x = \frac{x_2}{\lambda_2 - \lambda_1} \left[ \lambda_2 e^{\lambda_2 t} - \lambda_1 e^{\lambda_1 t} \right]
\]

For \(x = 0\): when \(t \neq \infty\), we must have

\[
\frac{\lambda_2}{\lambda_1} e^{\lambda_2 t} - \frac{\lambda_2}{\lambda_1} e^{\lambda_1 t} = 0
\]

(3)

Recall that

\[
\lambda_1 < 0, \quad \lambda_2 < 0.
\]

Choosing \(\lambda_1\) and \(\lambda_2\) so that \(\lambda_1 < \lambda_2 < 0\), we have

\[
0 < \frac{\lambda_2}{\lambda_1} < 1 \quad \text{and} \quad \lambda_2 - \lambda_1 > 0
\]

Thus a positive solution for \(t > 0\) for Equation (3) cannot exist, since it would require that \(e\) raised to a positive power be less than 1, which is impossible. Thus, \(x\) is never 0.

The \(x-t\) curve for this case is as shown.
PROBLEM 19.127 (Continued)

(b) \( t = 0, \ x = 0, \ v = v_0 \): Equations (1) and (2) yield

\[
0 = c_1 + c_2 \\
v_0 = c_1 \lambda_1 + c_2 \lambda_2
\]

Solving for \( c_1 \) and \( c_2 \),

\[
c_1 = -\frac{v_0}{\lambda_2 - \lambda_1} \\
c_2 = \frac{v_2}{\lambda_2 - \lambda_1}
\]

Substituting into Eq. (1),

\[
x = \frac{v_0}{\lambda_2 - \lambda_1} \left[ e^{\lambda_1 t} - e^{\lambda_2 t} \right]
\]

For \( x = 0 \),

\[
t \neq \alpha \\
e^{\lambda_2 t} = e^{\lambda_1 t}
\]

For \( c > c_c, \ \lambda_1 - \lambda_2 \); thus, no solution can exist for \( t \), and \( x \) is never 0.

The \( x-t \) curve for this case is as shown.
PROBLEM 19.128

Show that in the case of heavy damping \((c > c_c)\), a body released from an arbitrary position with an arbitrary initial velocity cannot pass more than once through its equilibrium position.

SOLUTION

Substitute the initial conditions, \(t = 0, x = x_0, v = v_0\) in Equations (1) and (2) of Problem 19.127.

\[
x_0 = c_1 + c_2, \quad v_0 = c_1\lambda_1 + c_2\lambda_2
\]

Solving for \(c_1\) and \(c_2\),

\[
c_1 = -\frac{(v_0 - \lambda_2x_0)}{\lambda_2 - \lambda_1}
\]

\[
c_2 = \frac{(v_0 - \lambda_4x_0)}{\lambda_2 - \lambda_1}
\]

And substituting in Eq. (1)

\[
x = \frac{1}{\lambda_2 - \lambda_1} \left[ (v_0 - \lambda_4x_0)e^{\lambda_4t} - (v_0 - \lambda_2x_0)e^{\lambda_2t} \right]
\]

For \(x = 0, t \neq \infty\):

\[
(v_0 - \lambda_4x_0)e^{\lambda_4t} = (v_0 - \lambda_2x_0)e^{\lambda_2t}
\]

\[
e^{(\lambda_2 - \lambda_4)t} = \frac{(v_0 - \lambda_2x_0)}{(v_0 - \lambda_4x_0)}
\]

\[
t = \frac{1}{(\lambda_2 - \lambda_4)} \ln \frac{v_0 - \lambda_2x_0}{v_0 - \lambda_4x_0}
\]

This defines one value of \(t\) only for \(x = 0\), which will exist if the natural log is positive, i.e., if \(\frac{v_0 - \lambda_2x_0}{v_0 - \lambda_4x_0} > 1\). Assuming \(\lambda_1 < \lambda_2 < 0\), this occurs if \(v_0 < \lambda_4x_0\).
PROBLEM 19.129

In the case of light damping, the displacements $x_1$, $x_2$, $x_3$, shown in Figure 19.11 may be assumed equal to the maximum displacements. Show that the ratio of any two successive maximum displacements $x_n$ and $x_{n+1}$ is constant and that the natural logarithm of this ratio, called the logarithmic decrement, is

$$\ln \frac{x_n}{x_{n+1}} = \frac{2\pi (c/c_c)}{\sqrt{1-(c/c_c)^2}}$$

SOLUTION

For light damping,

Equation (19.46):

$$x = x_0 e^{-\left(\frac{t}{\omega_D}\right)} \sin(\omega_D t + \phi)$$

At given maximum displacement,

$$t = t_n, \ x = x_n$$

$$\sin(\omega_D t_n + \phi) = 1$$

$$x_n = x_0 e^{-\left(\frac{t_n}{\omega_D}\right)}$$

At next maximum displacement,

$$t = t_{n+1}, \ x = x_{n+1}$$

$$\sin(\omega_D t_{n+1} + \phi) = 1$$

$$x_{n+1} = x_0 e^{-\left(\frac{t_{n+1}}{\omega_D}\right)}$$

But

$$\omega_D t_{n+1} - \omega_D t_n = 2\pi$$

$$t_{n+1} - t_n = \frac{2\pi}{\omega_D}$$

Ratio of successive displacements:

$$\frac{x_n}{x_{n+1}} = \frac{x_0 e^{-\frac{t_n}{\omega_D}}}{x_0 e^{-\frac{t_{n+1}}{\omega_D}}}$$

$$= e^{-\frac{2\pi}{\omega_D}(t_n - t_{n+1})} = e^{\frac{2\pi}{\omega_D}}$$

Thus,

$$\ln \frac{x_n}{x_{n+1}} = \frac{c\pi}{m\omega_D}$$

(1)

From Equations (19.45) and (19.41):

$$\omega_D = \omega_n \sqrt{1 - \left(\frac{c}{c_c}\right)^2}$$

$$\omega_D = \frac{c_c}{2m} \sqrt{1 - \left(\frac{c}{c_c}\right)^2}$$

Thus,

$$\ln \frac{x_n}{x_{n+1}} = \frac{c\pi}{m c_c} \frac{1}{\sqrt{1 - \left(\frac{c}{c_c}\right)^2}} \ln \frac{x_n}{x_{n+1}} = \frac{2\pi (c/c_c)}{\sqrt{1-(c/c_c)^2}}$$

Q.E.D.
PROBLEM 19.130

In practice, it is often difficult to determine the logarithmic decrement of a system with light damping defined in Problem 19.129 by measuring two successive maximum displacements. Show that the logarithmic decrement can also be expressed as \((1/k) \ln (x_n/x_{n+k})\), where \(k\) is the number of cycles between readings of the maximum displacement.

SOLUTION

As in Problem 19.129, for maximum displacements \(x_n\) and \(x_{n+k}\) at \(t_n\) and \(t_{n+k}\), \(\sin(\omega_0 t_n + \phi) = 1\)

and \(\sin(\omega_0 t_{n+k} + \phi) = 1\).

\[ x_n = x_0 e^{-\left(\frac{k}{2}\right) n} \]

\[ x_{n+k} = x_0 e^{-\left(\frac{k}{2}\right) (n+k)} \]

Ratio of maximum displacements:

\[ \frac{x_n}{x_{n+k}} = \frac{x_0 e^{\left(\frac{k}{2}\right) n}}{x_0 e^{\left(\frac{k}{2}\right) (n+k)}} = e^{\left(\frac{k}{2}\right) (t_n-t_{n+k})} \]

But

\[ \omega_D t_{n+k} - \omega_D t_n = k(2\pi) \]

\[ t_n - t_{n+k} = k \frac{2\pi}{\omega_D} \]

Thus,

\[ \frac{x_n}{x_{n+k}} = + \frac{c}{2m} \left( \frac{2k\pi}{\omega_D} \right) \]

\[ \ln \frac{x_n}{x_{n+k}} = k \frac{c\pi}{m\omega_D} \]

(2)

But from Problem 19.129, Equation (1):

\[ \log \text{ decrement} = \ln \frac{x_n}{x_{n+1}} = \frac{c\pi}{m\omega_D} \]

Comparing with Equation (2),

\[ \log \text{ decrement} = \frac{1}{k} \ln \frac{x_n}{x_{n+k}} \quad \text{Q.E.D.} \]
PROBLEM 19.131

In a system with light damping \((c < c_c)\), the period of vibration is commonly defined as the time interval \(\tau_d = \frac{2\pi}{\omega_d}\) corresponding to two successive points where the displacement-time curve touches one of the limiting curves shown in Figure 19.11. Show that the interval of time \((a)\) between a maximum positive displacement and the following maximum negative displacement is \(\frac{1}{2} \tau_d\), \((b)\) between two successive zero displacements is \(\frac{1}{2} \tau_d\), \((c)\) between a maximum positive displacement and the following zero displacement is greater than \(\frac{1}{4} \tau_d\).

SOLUTION

Equation (19.46):

\[ x = x_0 e^{-\left(\frac{c}{2m}\right)t} \sin(\omega_D t + \phi) \]

\((a)\) Maxima (positive or negative) when \(\dot{x} = 0:\n\)

\[ \dot{x} = x_0 \left( -\frac{c}{2m} \right) e^{-\left(\frac{c}{2m}\right)t} \sin(\omega_D t + \phi) + x_0 \omega_D e^{-\left(\frac{c}{2m}\right)t} \cos(\omega_D t + \phi) \]

Thus, zero velocities occur at times when

\[ \dot{x} = 0, \quad \text{or} \quad \tan(\omega_D t + \phi) = \frac{2m\omega_D}{c} \tag{1} \]

The time to the first zero velocity, \(t_1\), is

\[ t_1 = \frac{\tan^{-1}\left(\frac{2m\omega_D}{c}\right) - \phi}{\omega_D} \tag{2} \]

The time to the next zero velocity where the displacement is negative is

\[ t_1' = \frac{\tan^{-1}\left(\frac{2m\omega_D}{c}\right) - \phi + \pi}{\omega_D} \tag{3} \]
PROBLEM 19.131 (Continued)

Subtracting Eq. (2) from Eq. (3),

\[ t'_1 - t_1 = \frac{\pi}{\omega_D} = \frac{\pi \cdot \tau_D}{2\pi} = \frac{\tau_D}{2} \quad \text{Q.E.D.} \]

(b) Zero displacements occur when

\[ \sin(\omega_D t + \phi) = 0 \quad \text{or at intervals of} \]

\[ \omega_D t + \phi = \pi, \ 2\pi \ n\pi \]

Thus,

\[ \frac{(t_1)_0}{\omega_D} = (\pi - \phi) \quad \text{and} \quad \frac{(t'_1)_0}{\omega_D} = \frac{(2\pi - \phi)}{\omega_D} \]

Time between

\[ 0'_1 = (t'_1)_0 - (t_1)_0 = \frac{2\pi - \phi}{\omega_D} = \frac{\pi \tau_D}{2\pi} = \frac{\tau_D}{2} \quad \text{Q.E.D.} \]

Plot of Equation (1)

(c) The first maxima occurs at 1:

\[ (\omega_D t_1 + \phi) \]

The first zero occurs at

\[ (\omega_D (t_1)_0 + \phi) = \pi \]

From the above plot,

\[ (\omega_D (t_1)_0 + \phi) - (\omega_D t'_1 + \phi) > \frac{\pi}{2} \]

or

\[ (t_1)_0 - t_1 > \frac{\pi}{2\omega_D} \quad (t'_1)_0 - t_1 > \frac{\tau_D}{4} \quad \text{Q.E.D.} \]

Similar proofs can be made for subsequent maximum and minimum.
**PROBLEM 19.132**

The block shown is depressed 30 mm from its equilibrium position and released. Knowing that after 10 cycles the maximum displacement of the block is 1.25 mm, determine (a) the damping factor \( \frac{c}{c_c} \), (b) the value of the coefficient of viscous damping. (*Hint: See Problems 19.129 and 19.130.*)

**SOLUTION**

From Problems 19.130 and 19.129:

\[
\left(\frac{1}{k}\right) \ln \left(\frac{x_n}{x_n + k}\right) = \frac{2\pi \frac{c}{c_c}}{\sqrt{1 - \left(\frac{c}{c_c}\right)^2}}
\]

where \( k = \) number of cycles = 10

(a) First maximum is \( x_1 = 30 \text{ mm} \)

Thus, \( n = 1 \)

\[
\frac{x_1}{x_1 + 10} = \frac{30}{1.25} = 2.4
\]

\[
\frac{1}{10} \ln 2.4 = 0.08755
\]

\[
= \frac{2\pi \frac{c}{c_c}}{\sqrt{1 - \left(\frac{c}{c_c}\right)^2}}
\]

Damping factor:

\[
1 - \left(\frac{c}{c_c}\right)^2 = \left(\frac{2\pi}{0.08755}\right)^2 \left(\frac{c}{c_c}\right)^2
\]

\[
\left(\frac{c}{c_c}\right)^2 \left[\left(\frac{2\pi}{0.08755}\right)^2 + 1\right] = 1
\]

\[
\left(\frac{c}{c_c}\right)^2 = \frac{1}{(5150 + 1)}
\]

\[
= 0.0001941
\]

\[
\frac{c}{c_c} = 0.01393
\]

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PROBLEM 19.132 (Continued)

(b) Critical damping coefficient. \[ c_c = 2 m \sqrt{\frac{k}{m}} \] (Eq. 19.41)

or

\[ c_c = 2\sqrt{km} \]
\[ c_c = 2\sqrt{(175 \text{ N/m})(4 \text{ kg})} \]
\[ c_c = 52.915 \text{ N\cdot s/m} \]

From Part (a),

\[ \frac{c}{c_c} = 0.01393 \]
\[ c = (0.01393)(52.915) \]

Coefficient of viscous damping.

\[ c = 0.737 \text{ N\cdot s/m \phantom{1}} \]
**PROBLEM 19.133**

A loaded railroad car of mass 15,000 kg is rolling at a constant velocity $v_0$ when it couples with a spring and dashpot bumper system (Figure 1). The recorded displacement-time curve of the loaded railroad car after coupling is as shown (Figure 2). Determine $(a)$ the damping constant, $(b)$ the spring constant. (Hint: Use the definition of logarithmic decrement given in Problem 19.129.)

**SOLUTION**

Mass of railroad car: $m = 15000$ kg

The differential equation of motion for the system is

$$m\ddot{x} + c\dot{x} + kx = 0$$

For light damping, the solution is given by Eq. (19.44):

$$x = e^{-(\frac{\omega_d}{2})t}(c_1 \sin \omega_d t + c_2 \cos \omega_d t)$$

From the displacement versus time curve,

$$\tau_d = 0.41 \text{ s}$$

$$\omega_d = \frac{2\pi}{\tau_d} = \frac{2\pi}{0.41} = 15.325 \text{ rad/s}$$

At the first peak, $x_1 = 12.5 \text{ mm}$ and $t = t_1$.

At the second peak, $x_2 = 3 \text{ mm}$ and $t = t_1 + \tau_d$.

Forming the ratio $\frac{x_2}{x_1}$,

$$\frac{x_2}{x_1} = \frac{e^{-\left(\frac{\omega_d}{2}\right)(t_1+\tau_d)}}{e^{-\left(\frac{\omega_d}{2}\right)t_1}} = e^{-\left(\frac{\omega_d}{2}\right)\tau_d}$$

$$\frac{x_1}{x_2} = e^{\frac{\omega_d}{\tau_d}}$$
PROBLEM 19.133 (Continued)

(a) **Damping constant.**

From Eq. (1):

\[
\frac{ct \tau_d}{2m} = \ln \left( \frac{x_1}{x_2} \right)
\]

\[
c = \frac{2m}{\tau_d} \ln \frac{x_1}{x_2} = \frac{(2)(15000)}{0.41} \ln \frac{12.5}{3} = 104.423 \times 10^3 \text{ N} \cdot \text{s/m}
\]

\[c = 104.4 \text{ kN} \cdot \text{s/m} \]

(b) **Spring constant.**

Equation for \(\omega_d\):

\[
\omega_d^2 = \frac{k}{m} - \left( \frac{c}{2m} \right)^2
\]

\[k = m\omega_d^2 + \frac{c^2}{4m} = (15000)(15.325)^2 + \frac{(104.423 \times 10^3)^2}{(4)(15000)}
\]

\[= 3.7046 \times 10^6 \text{ N/m} \quad k = 3.70 \times 10^6 \text{ N/m} \]
PROBLEM 19.134

A 4-kg block $A$ is dropped from a height of 800 mm onto a 9-kg block $B$ which is at rest. Block $B$ is supported by a spring of constant $k = 1500 \text{ N/m}$ and is attached to a dashpot of damping coefficient $c = 230 \text{ N} \cdot \text{s/m}$. Knowing that there is no rebound, determine the maximum distance the blocks will move after the impact.

SOLUTION

Velocity of Block $A$ just before impact.

$$v_A = \sqrt{2gh} = \sqrt{2(9.81)(0.8)} = 3.962 \text{ m/s}$$

Velocity of Blocks $A$ and $B$ immediately after impact.

Conservation of momentum.

$$m_Av_A + m_Bv_B = (m_A + m_B)v'$$

$(4)(3.962) + 0 = (4 + 9)v'$

$$v' = 1.219 \text{ m/s}$$

$$\ddot{x}_0 = +1.219 \text{ m/s} \downarrow = \dot{x}_0$$

Static deflection (Block $A$):

$$x_0 = -\frac{m_Ag}{k} = -\frac{(4)(9.82)}{1500} = -0.02619 \text{ m}$$

$x = 0$, Equivalent position for both blocks:

$$c_c = 2\sqrt{km} = 2\sqrt{(1500)(13)} = 279.3 \text{ N} \cdot \text{s/m}$$

Since $c < c_c$, Equation (19.44):

$$x = e^{-\left(\frac{t}{c_c}\right)}\left[c_1 \sin \omega_D t + c_2 \cos \omega_D t\right]$$

$$\frac{c}{2m} = \frac{230}{2(13)} = 8.846 \text{ s}^{-1}$$
PROBLEM 19.134 (Continued)

Expression for $\omega_D$:

$$\omega_D^2 = \frac{k}{m} - \left( \frac{c}{2m} \right)^2$$

$$\omega_D = \sqrt{\frac{1500}{13} - \left( \frac{230}{2(13)} \right)^2}$$

$$= 6.094 \text{ rad/s}$$

$$x = e^{-8.846t} (c_1 \sin 6.094t + c_2 \cos 6.094t)$$

Initial conditions:

$$x_0 = -0.02619 \text{ m}$$

$$\dot{x}_0 = +1.219 \text{ m/s}$$

$$x_0 = -0.02619 = e^0 [c_1(0) + c_2(1)]$$

$$c_2 = -0.02619$$

$$\dot{x}(0) = -8.846e^{-8.846} [c_1(0) + (-0.02619)(1)]$$

$$+ e^{-8.846} [6.094c_1(1) + c_2(0)] = 1.219$$

$$1.219 = (-8.846)(-0.02619) + 6.094c_1$$

$$c_1 = 0.16202$$

$$x = e^{-8.846t} (0.16202 \sin 6.094t - 0.02619 \cos 6.094t)$$

Maximum deflection occurs when $\dot{x} = 0$

$$\dot{x} = 0 = -8.846e^{-8.846\tau_m} (0.16202 \sin 6.094\tau_m - 0.02619 \cos 6.094\tau_m)$$

$$+ e^{-8.846\tau_m} [6.094][0.1620 \cos 6.094\tau_m + 0.02619 \sin 6.094\tau_m]$$

$$0 = [(-8.846)(0.16202) + (6.094)(0.02619)] \sin 6.094\tau_m$$

$$+ [(-8.846)(-0.02619) + (6.094)(0.1620)] \cos 6.094\tau_m$$

$$0 = -1.274 \sin 6.094t + 1.219 \cos 6.094t$$

$$\tan 6.094t = \frac{1.219}{1.274} = 0.957$$

$$\tan^{-1} 0.957 = 0.1253 \text{ s}$$

$$\tau_m = \tan^{-1} \frac{0.957}{6.094} = 0.1253 \text{ s}$$

$$x_m = e^{-[(8.846)(0.1253)]} [0.1620 \sin (6.094)(0.1253)]$$

$$- 0.02619 \cos (6.094)(0.1253)]$$

$$x_m = (0.3301)(0.1120 - 0.0189) = 0.307 \text{ m}$$

Blocks move, static deflection $+ x_m$  

Total distance $= 0.02619 + 0.307 = 0.0569 \text{ m} = 56.9 \text{ mm}$
PROBLEM 19.135

Solve Problem 19.134, assuming that the damping coefficient of the dashpot is $c = 300 \text{ N} \cdot \text{s/m}$.

SOLUTION

Velocity of Block $A$ just before impact.

$$v_A = \sqrt{2gh}$$
$$= \sqrt{2(9.81)(0.8)}$$
$$= 3.962 \text{ m/s}$$

Velocity of Blocks $A$ and $B$ immediately after impact.

Conservation of momentum.

$$m_Av_A + m_Bv_B = (m_A + m_B)v'$$
$$4(3.962) + 0 = (4 + 9)v'$$
$$v' = 1.219 \text{ m/s}$$
$$\dot{x}_0 = +1.219 \text{ m/s} \downarrow = \dot{x}_0$$

Static deflection (Block $A$).

$$x_0 = -\frac{m_Ag}{k}$$
$$= -\frac{(4)(9.82)}{1500}$$
$$= -0.02619 \text{ m}$$

$x = 0$, Equivalent position for both blocks:

$$c_c = 2\sqrt{k}m$$
$$= 2\sqrt{(1500)(13)}$$
$$= 279.3 \text{ N} \cdot \text{s/m}$$

Since $c > c_c$, the system is heavily damped.

Eq. 19.42:

$$x = c_1e^{\lambda t} + c_2e^{\lambda t}$$
PROBLEM 19.135 (Continued)

Eq. 19.40:

\[
\lambda = \frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}
\]

\[
= \frac{-300}{26} \pm \sqrt{\left(\frac{-300}{26}\right)^2 - \frac{1500}{13}}
\]

\[
= -11.538 \pm 4.213
\]

\[
\lambda_1 = -15.751 \text{ s}^{-1}
\]

\[
\lambda_2 = -7.325 \text{ s}^{-1}
\]

\[
x = c_1 e^{-15.751t} + c_2 e^{-7.325t}
\]

\[
\dot{x} = -15.751c_1 e^{-11.751t} - 7.325c_2 e^{-7.325t}
\]

Initial conditions:

\[
x_0 = -0.02619 \text{ m}
\]

\[
\dot{x}_0 = 1.219 \text{ m/s}
\]

Then

\[
-0.02619 = c_1 + c_2
\]

\[
1.219 = -15.751c_1 - 7.325c_2
\]

Solving the simultaneous equations,

\[
c_1 = -0.1219 \text{ m}
\]

\[
c_2 = 0.09571 \text{ m}
\]

Then

\[
x = -0.1219 e^{-15.751t} + 0.09571 e^{-7.325t} \text{ (m)}
\]

\[
\dot{x} = (15.751)(-0.1219) e^{-15.751t} - (7.325)(0.09571) e^{-7.325t}
\]

\[
= 1.92 e^{-15.751t} - 0.70108 e^{-7.325t}
\]

The maximum value of \(x\) occurs when \(\dot{x} = 0\)

\[
0 = 1.92 e^{-15.751t_{m}} - 0.70108 e^{-7.325t_{m}}
\]

\[
1.92 e^{-15.751t_{m}} = 0.70108 e^{-7.325t_{m}}
\]

\[
\frac{1.92}{0.70108} = e^{(15.751 - 0.7325)t_{m}}
\]

\[
2.7386 = e^{8.426t_{m}}
\]

\[
8.426t_{m} = \ln(2.7386)
\]

\[
t_{m} = 0.11956 \text{ s}
\]

\[
x_{m} = -0.1219 e^{-(15.751)(0.11956)} + 0.09571 e^{-(7.325)(0.11956)}
\]

\[
= -0.018542 + 0.039867
\]

\[
= 0.02132 \text{ m}
\]

Total deflection = static deflection + \(x_{m}\)

\[
= 0.02619 + 0.02132
\]

\[
= 0.04751 \text{ m}
\]

Total deflection = 47.5 mm
PROBLEM 19.136

The barrel of a field gun weighs 750 kg and is returned into firing position after recoil by a recuperator of constant \( c = 18 \text{ kN} \cdot \text{s/m} \). Determine (a) the constant \( k \) which should be used for the recuperator to return the barrel into firing position in the shortest possible time without any oscillation, (b) the time needed for the barrel to move back two-thirds of the way from its maximum-recoil position to its firing position.

SOLUTION

(a) A critically damped system regains its equilibrium position in the shortest time.

\[ c = c_c \\
= 18000 \\
= 2m \sqrt{\frac{k}{m}} \]

Then

\[ k = \frac{(c)^2}{m} = \frac{(18000)^2}{750} = 108 \text{ kN/m} \]

(b) For a critically damped system, Equation (19.43):

\[ x = (c_1 + c_2 t)e^{-\omega_n t} \]

We take \( t = 0 \) at maximum deflection \( x_0 \).

Thus,

\[ \dot{x}(0) = 0 \]

\[ x(0) = x_0 \]

Using the initial conditions,

\[ x(0) = x_0 = (c_1 + 0)e^{0}, \quad \text{so} \quad c_1 = x_0 \]

\[ x = (x_0 + c_2 t)e^{-\omega_n t} \]

and

\[ \dot{x} = -\omega_n (x_0 + c_2 t)e^{-\omega_n t} + c_2 e^{-\omega_n t} \]

\[ \dot{x}(0) = 0 = -\omega_n x_0 + c_2, \quad \text{so} \quad c_2 = \omega_n x_0 \]

Thus,

\[ x = x_0 (1 + \omega_n t)e^{-\omega_n t} \]

For

\[ x = \frac{x_0}{3}, \quad \frac{1}{3} = (1 + \omega_n t)e^{-\omega_n t}, \quad \text{with} \quad \omega_n = \sqrt{\frac{k}{m}} \quad (1) \]

We obtain

\[ \omega_n = \sqrt{\frac{108 \times 10^3}{750}} = 12 \text{ rad/s}^{-1} \]

From solution of (1)

\[ \omega_n t = 2.289, \]

\[ t = \frac{2.289}{12} = 0.19075 \text{ s} \quad t = 0.1908 \text{ s} \]
PROBLEM 19.137

A uniform rod of mass $m$ is supported by a pin at $A$ and a spring of constant $k$ at $B$ and is connected at $D$ to a dashpot of damping coefficient $c$. Determine in terms of $m$, $k$, and $c$, for small oscillations, (a) the differential equation of motion, (b) the critical damping coefficient $c_c$.

SOLUTION

In equilibrium, the force in the spring is $mg$.

For small angles, $\sin \theta = \theta \cos \theta = 1$

$\delta y_B = \frac{l}{2} \theta$

$\delta y_C = \frac{l}{2} \bar{\theta}$

(a) Newton’s Law:

$\Sigma M_A = (\Sigma M_A)_{\text{eff}}$

$\left( \frac{mg}{2} \right) - \left( \frac{k}{2} \theta + mg \right) \frac{l}{2} - c \bar{\theta} l = \ddot{\theta} + m \ddot{\bar{\theta}} l \frac{l}{2}$

Kinematics:

$\alpha = \ddot{\theta}$

$\bar{\alpha} = \frac{l}{2} \alpha = \frac{l}{2} \ddot{\theta}$

$\left[ \bar{T} + m \left( \frac{l}{2} \right)^2 \right] \ddot{\theta} + c \dot{\theta} \dot{\theta} + k \left( \frac{l}{2} \right)^2 \theta = 0$

$\bar{T} + m \left( \frac{l}{2} \right)^2 = \frac{1}{3} ml^2$

$\ddot{\theta} + \left( \frac{3c m}{m} \right) \dot{\theta} + \left( \frac{3k m}{4m} \right) \theta = 0$

(b) Substituting $\theta = e^{\lambda t}$ into the differential equation obtained in (a), we obtain the characteristic equation,

$\lambda^2 + \left( \frac{3c m}{m} \right) \lambda + \frac{3k m}{4m} = 0$

and obtain the roots

$\lambda = \frac{-3c m + \sqrt{(3c m)^2 - (3k m)^2}}{2}$

The critical damping coefficient, $c_c$, is the value of $c$, for which the radicand is zero.

Thus,

$\left( \frac{3c_c m}{m} \right)^2 = \frac{3k m}{4m}$

$c_c = \sqrt{\frac{km}{3}}$
PROBLEM 19.138

A 2-kg uniform rod is supported by a pin at $O$ and a spring at $A$, and is connected to a dashpot at $B$. Determine (a) the differential equation of motion for small oscillations, (b) the angle that the rod will form with the horizontal 5 s after end $B$ has been pushed 25 mm down and released.

SOLUTION

Small angles: 
\[
\sin \theta = \theta, \quad \cos \theta = 1 \\
\delta y_A = (0.2 \text{ m})\theta = 0.2 \theta \\
\delta y_C = (0.2 \text{ m})\theta = 0.2 \theta \\
\delta y_B = (0.6 \text{ m})\theta = 0.6 \theta
\]

(a) Newton’s Law:

\[
\Sigma M_0 = (\Sigma M_0)_{\text{eff}} + \gamma - (0.2 \text{ m})F_x + (0.2 \text{ m})(2)(9.81) - (0.6 \text{ m})F_D \\
= T\alpha + (0.2 \text{ m})m\ddot{\alpha}
\]

\[
F_x = k(\delta y_A + (\delta_{ST})_A) = k(0.2 \theta + (\delta_{ST})_A) \\
F_D = c\delta y_B = c0.6 \dot{\theta} \\
T = \frac{1}{12}ml^2 = \frac{1}{12}m(0.8)^2 = \frac{4 \text{ m}}{75}
\]

Kinematics:

\[\alpha = \dot{\theta}, \quad \ddot{\alpha} = (0.2 \text{ m})\alpha = 0.2 \dot{\theta}\]

Thus, from Eq. (1),

\[
\left[\frac{4m}{75} + (0.2)^2 m\right]\ddot{\theta} + (0.6)^2 c \dot{\theta} + 0.2k((0.2 \theta + (\delta_{ST})_A) - (0.2)(2)(9.81) = 0
\]

But in equilibrium,

\[
\Sigma M_0 = 0 \\
\gamma k(\delta_{ST})_A(0.2) - (2)(9.81)(0.2) = 0, \quad 0.2k(\delta_{ST})_A = (0.2)(2)(9.81)
PROBLEM 19.138 (Continued)

Equation (2) becomes
\[
\left( \frac{7}{75} \right) m \ddot{\theta} + \left( \frac{9}{25} c \right) \left( \frac{1}{25} k \theta \right) = 0
\]
\[
\frac{7}{75} m = \left( \frac{7}{75} \right) (2) = 0.18667
\]
\[
\frac{9}{25} c = \left( \frac{9}{25} \right) (8) = 2.88
\]
\[
\frac{k}{25} = \frac{50}{25} = 2
\]
\[
0.18667 \ddot{\theta} + 2.88 \dot{\theta} + 2 \theta = 0
\]

(b) Substituting \( e^{\lambda t} \) into the above differential equation,
\[
0.18667 \lambda^2 + 2.88 \lambda + 2 = 0
\]
\[
\lambda = -0.7289, -14.699i
\]
Since the roots are negative and real, the system is heavily damped. (Eq. 19.46):
\[
\theta = c_1 e^{-0.7289t} + c_2 e^{-14.699t}
\]
\[
\dot{\theta} = -(0.7289c_1)e^{-0.7289t} - (14.699c_2)e^{-14.699t}
\]
Initial conditions.
\[
\theta_0 = \sin^{-1} \frac{25}{600} = 0.04168 \text{ rad}
\]
\[
\theta_0 = 0
\]
\[
0.04168 = c_1 + c_2
\]
\[
0 = -0.07289c_1 - 14.699c_2
\]
Solving for \( c_1 \) and \( c_2 \):
\[
c_1 = 0.04385, c_2 = -2.1747 \times 10^{-3}
\]
At \( t = 5 \),
\[
\theta = 0.04385e^{-0.7289(5)} - 2.1747 \times 10^{-3} e
\]
\[
= 0.1146 \text{ rad} = 65.7^\circ
\]
PROBLEM 19.139

A 500-kg machine element is supported by two springs, each of constant 50 kN/m. A periodic force of 150 N amplitude is applied to the element with a frequency of 2.8 Hz. Knowing that the coefficient of damping is 1.8 kN·s/m, determine the amplitude of the steady-state vibration of the element.

SOLUTION

Eq. (19.52):

\[ x_m = \frac{P_m}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} \]

Total spring constant:

\[ k = (2)(50 \text{ kN/m}) = 100 \times 10^3 \text{ N/m} \]

\[ \omega_f = 2\pi f_f = 2\pi(2.8) = 5.6\pi \text{ rad/s} \]

\[ m = 500 \text{ kg} \]

\[ P_m = 150 \text{ N} \]

\[ c = 1800 \text{ N} \cdot \text{s/m} \]

\[ x_m = \frac{150}{\sqrt{[100,000 - (500)(5.6\pi)^2]^2 + [1800(5.6\pi)^2]^2}} \]

\[ = \frac{150}{\sqrt{2.9982 \times 10^9 + 1.0028 \times 10^9}} \]

\[ = 2.3714 \times 10^{-3} \text{ m} = 2.3714 \text{ mm} \]

\[ x_m = 2.37 \text{ mm} \]
PROBLEM 19.140

In Problem 19.139, determine the required value of the constant of each spring if the amplitude of the steady-state vibration is to be 1.25 mm.

SOLUTION

\[ \omega_f = 2\pi f_f = (2\pi)(2.8) = 5.6\pi \text{ rad/s} \]
\[ P_m = 150 \text{ N} \]
\[ m = \frac{W}{g} = 500 \text{ kg} \]
\[ c = 1800 \text{ N}\cdot\text{s/m} \]
\[ x_m = 1.25 \text{ mm} \times 1.25 \times 10^{-3} \text{ m} \]

Eq. (19.52):

\[ x_m = \frac{P_m}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} \]

Solve for \( k \):

\[ (k - m\omega_f^2)^2 + (c\omega_f)^2 = \left(\frac{P_m}{x_m}\right)^2 \]

\[ k = m\omega_f^2 + \sqrt{\left(\frac{P_m}{x_m}\right)^2 - (c\omega_f)^2} \]

\[ = (500)(5.6\pi)^2 + \sqrt{\left(\frac{150}{1.25 \times 10^{-3}}\right)^2 - [(1800)(5.6\pi)]^2} \]
\[ = 154.755 \times 10^3 + \sqrt{1.44 \times 10^{10} - 1.00281 \times 10^9} \]
\[ = 154.755 \times 10^3 + 115.746 \times 10^3 \]
\[ = 270.501 \times 10^3 \text{ N/m} \]

The above calculated value is the equivalent spring constant for two springs.

For each spring, the spring constant is \( k \):

\[ \frac{k}{2} = 135.3 \text{ kN/m} \]
PROBLEM 19.141

In the case of the forced vibration of a system, determine the range of values of the damping factor $c/c_c$ for which the magnification factor will always decrease as the frequency ratio $\omega_f/\omega_n$ increases.

SOLUTION

From Eq. (19.53):

Magnification factor:

$$\frac{x_m}{P_m} = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega_f}{\omega_n}\right)^2\right]^2 + \left[2 \left(\frac{c}{c_c}\right) \frac{\omega_f}{\omega_n}\right]^2}}$$

Find value of $\frac{c}{c_c}$ for which there is no maximum for $\frac{x_m}{P_m}$ as $\frac{\omega_f}{\omega_n}$ increases.

$$\frac{d}{d\left(\frac{\omega_f}{\omega_n}\right)} \left(\frac{x_m}{P_m}\right)^2 = -2 \left[1 - \left(\frac{\omega_f}{\omega_n}\right)^2\right] \left(-1 + 4 \frac{c^2}{c_c^2}\right)$$

$$\frac{d}{d\left(\frac{\omega_f}{\omega_n}\right)} \left(\frac{\omega_f}{\omega_n}\right)^2 = \left\{\left[1 - \left(\frac{\omega_f}{\omega_n}\right)^2\right]^2 + \left[2 \left(\frac{c}{c_c}\right) \frac{\omega_f}{\omega_n}\right]^2\right\}$$

$$-2 + 2 \left(\frac{\omega_f}{\omega_n}\right)^2 + 4 \frac{c^2}{c_c^2} = 0$$

$$\left(\frac{\omega_f}{\omega_n}\right)^2 = 1 - 2 \frac{c^2}{c_c^2}$$

For $\frac{c^2}{c_c^2} \geq \frac{1}{2}$, there is no maximum for $\frac{x_m}{P_m}$ and the magnification factor will decrease as $\frac{\omega_f}{\omega_n}$ increases.

$$\frac{c}{c_c} \geq \frac{1}{\sqrt{2}}$$

$$\frac{c}{c_c} \geq 0.707$$
**PROBLEM 19.142**

Show that for a small value of the damping factor $c/c_c$, the maximum amplitude of a forced vibration occurs when $\omega_f = \omega_n$ and that the corresponding value of the magnification factor is $\frac{1}{2}(c/c_c)$.

**SOLUTION**

From Eq. (19.53'):

\[
\text{Magnification factor} = \frac{x_m}{\frac{P_c}{k}} = \frac{1}{\sqrt{1 - \left(\frac{\omega_f}{\omega_n}\right)^2 + \left[2 \left(\frac{c}{c_c}\right) \left(\frac{\omega_f}{\omega_n}\right)\right]^2}}
\]

Find value of $\frac{\omega_f}{\omega_n}$ for which $\frac{x_m}{\frac{P_c}{k}}$ is a maximum.

\[
0 = d\left(\frac{x_m}{\frac{P_c}{k}}\right)^2 = -2 \left[1 - \left(\frac{\omega_f}{\omega_n}\right)^2\right] (1) + 4 \left(\frac{c}{c_c}\right) \left(\frac{\omega_f}{\omega_n}\right)^2
\]

\[
-2 + 2 \left(\frac{\omega_f}{\omega_n}\right)^2 + 4 \left(\frac{c}{c_c}\right)^2 = 0
\]

For small $\frac{c}{c_c}$,

\[
\frac{\omega_f}{\omega_n} = 1 \quad \omega_f = \omega_n
\]

For

\[
\frac{\omega_f}{\omega_n} = 1
\]

\[
\frac{x_m}{\frac{P_c}{k}} = \frac{1}{\sqrt{1 - 1 + \frac{\left(\frac{c}{c_c}\right)^2}{4}}}
\]

\[
\frac{x_m}{\frac{P_c}{k}} = \frac{1}{2} \frac{c}{c_c}
\]
PROBLEM 19.143

A 50-kg motor is directly supported by a light horizontal beam, which has a static deflection of 6 mm due to the mass of the motor. The unbalance of the rotor is equivalent to a mass of 100 g located 75 mm from the axis of rotation. Knowing that the amplitude of the vibration of the motor is 0.8 mm at a speed of 400 rpm, determine (a) the damping factor $c/c_c$, (b) the coefficient of damping $c$.

SOLUTION

Spring constant:

$$k = \frac{m}{\delta_{ST}} = \frac{mg}{\delta_{ST}} = \frac{(50)(9.81)}{6 \times 10^{-3}} = 81.75 \times 10^3 \text{ N \cdot m}$$

Natural undamped circular frequency:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{81.75 \times 10^3}{50}} = 40.435 \text{ rad/s}$$

Unbalance:

$$m' = 100 \text{ g} = 0.100 \text{ kg}$$

$$r = 75 \text{ mm} = 0.075 \text{ m}$$

Forcing frequency:

$$\omega_f = 400 \text{ rpm} = 41.888 \text{ rad/s}$$

Unbalance force:

$$P_m = m' r \omega_f^2 = (0.100)(0.075)(41.888)^2 = 13.1595 \text{ N}$$

Static deflection:

$$\delta_{ST} = \frac{P_m}{k} = \frac{13.1595}{81.75 \times 10^3} = 0.16097 \times 10^{-3} \text{ m}$$

Amplitude:

$$x_m = 0.8 \text{ mm} = 0.8 \times 10^{-3} \text{ m}$$

Frequency ratio:

$$\frac{\omega_f}{\omega_n} = \frac{41.888}{40.435} = 1.0359$$

Eq. (19.53):

$$x_m = \frac{\delta_{ST}}{\sqrt{1 - \left(\frac{\omega_f}{\omega_n}\right)^2 + \left[2 \left(\frac{c}{c_c}\right) \left(\frac{\omega_f}{\omega_n}\right)\right]^2}}$$

$$\left[1 - (1.0359)^2\right] + \left[2 \left(\frac{c}{c_c}\right) (1.0359)\right] = \left[0.16097 \times 10^{-3}\right]^2$$

$$0.0053523 + 4.2924 \left(\frac{c}{c_c}\right)^2 = 0.040486$$

$$\left(\frac{c}{c_c}\right)^2 = 0.008185$$
PROBLEM 19.143 (Continued)

(a) Damping factor.  \[ \frac{c}{c_c} = 0.090472 \quad \frac{c}{c_c} = 0.0905 \]

Critical damping factor.  \[ c_c = 2\sqrt{km} \]
\[ = 2\sqrt{(81.75 \times 10^4)(50)} \]
\[ = 4.0435 \times 10^3 \text{ N} \cdot \text{s/m} \]

(b) Coefficient of damping.  \[ c = \left( \frac{c}{c_c} \right) c_c \]
\[ = (0.090472)(4.0435 \times 10^3) \quad c = 366 \text{ N} \cdot \text{s/m} \]
PROBLEM 19.144

A 15-kg motor is supported by four springs, each of constant 45 kN/m. The unbalance of the motor is equivalent to a mass of 20 g located 125 mm from the axis of rotation. Knowing that the motor is constrained to move vertically, determine the amplitude of the steady-state vibration of the motor at a speed of 1500 rpm, assuming (a) that no damping is present, (b) that the damping factor $c/c_c$ is equal to 1.3.

SOLUTION

Mass of motor: $m = 15$ kg

Equivalent spring constant: $k = (4)(45 \times 10^3)$

$= 180 \times 10^3$ lb/ft

Undamped natural circular frequency: $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{180 \times 10^3}{15}}$

$= 109.545$ rad/s

Forcing frequency: $\omega_f = 1500$ rpm $= 157.08$ rad/s

Frequency ratio: $\frac{\omega_f}{\omega_n} = \frac{157.08}{109.545} = 1.43394$

Unbalance: $m' = 20 g = 20 \times 10^{-3}$ kg

$r = 125$ mm $= 0.125$ m

Unbalance force: $P_m = m'r\omega_f^2 = (20 \times 10^{-3})(0.125)(157.08)^2 = 61.685$ N

Static deflection: $\delta_{ST} = \frac{P_m}{k} = \frac{61.685}{180 \times 10^3}$

$= 0.3427 \times 10^{-3}$ m

Eq. (19.53):

$$x_m = \frac{\delta_{ST}}{\sqrt{1 - \left(\frac{\omega_f}{\omega_n}\right)^2} + \left[2\left(\frac{c}{c_c}\right)\frac{\omega_f}{\omega_n}\right]^2}}$$

(a) No damping:

$$1 - \left(\frac{\omega_f}{\omega_n}\right)^2 = [1 - (1.43394)^2]^2 = 1.11522$$

$$x_m = \frac{\delta_{ST}}{\sqrt{1.11522}}$$

$$= \frac{0.3427 \times 10^{-3}}{1.05618}$$

$$= 0.324 \times 10^{-3}$$ m

$x_m = 0.324$ mm \(\uparrow\)
PROBLEM 19.144 (Continued)

(b)

\[
\frac{c}{c_c} = 1.3 \left[ 1 - \left( \frac{\omega_c}{\omega_k} \right)^2 \right]^2 + \left[ 2 \left( \frac{\varepsilon}{\omega_c} \right) \left( \frac{\alpha_c}{\omega_k} \right) \right]^2 \\
= 1.11522 + [(2)(1.3)(1.43394)]^2 = 15.015
\]

\[
x_m = \frac{\delta_{ST}}{\sqrt{15.015}} = \frac{0.3427 \times 10^{-3}}{3.8749} \\
= 0.0884 \times 10^{-3} \text{ m} \\
x_m = 0.0884 \text{ mm} \downarrow
PROBLEM 19.145

A 100-kg motor is supported by four springs, each of constant 90 kN/m, and is connected to the ground by a dashpot having a coefficient of damping \( c = 6500 \text{ N} \cdot \text{s/m} \). The motor is constrained to move vertically, and the amplitude of its motion is observed to be 2.1 mm at a speed of 1200 rpm. Knowing that the mass of the rotor is 15 kg, determine the distance between the mass center of the rotor and the axis of the shaft.

SOLUTION

Equation (19.52):

\[
x_m = \frac{P_m}{\sqrt{\left(k - m\omega_f^2\right)^2 + c\omega_f^2}}
\]

\[
\omega_f^2 = \left(\frac{(1200)(2\pi)}{60}\right)^2
\]

\[
\omega_f^2 = 15,791 \text{ s}^{-2}
\]

\[
k = 4(90 \times 10^3) \text{ N/m}
\]

\[
= 360 \times 10^3
\]

\[
P_m = m'\omega_f^2
\]

\[
= (15 \text{ kg})e(15,791 \text{ s}^{-2})
\]

\[
= 236,865e
\]

\[
2.1 \times 10^{-3} m = \frac{236,865e}{\sqrt{\left[(360 \times 10^3) - (100)(15,791)\right]^2 + (6500)^2(15,791)}}
\]

\[
= 0.16141e
\]

\[
e = 13.01 \times 10^{-3} \text{ m}
\]

\[e = 13.01 \text{ mm} \uparrow\]
PROBLEM 19.146

A counter-rotating eccentric mass exciter consisting of two rotating 400-g masses describing circles of 150-mm radius at the same speed, but in opposite senses, is placed on a machine element to induce a steady-state vibration of the element and to determine some of the dynamic characteristics of the element. At a speed of 1200 rpm, a stroboscope shows the eccentric masses to be exactly under their respective axes of rotation and the element to be passing through its position of static equilibrium. Knowing that the amplitude of the motion of the element at that speed is 15 mm, and that the total mass of the system is 140 kg, determine (a) the combined spring constant $k$, (b) the damping factor $c/c_r$.

SOLUTION

Forcing frequency: \[ \omega_f = 1200 \text{ rpm} = 125.664 \text{ rad/s} \]

Unbalance of one mass: \[ m = 400 \text{ g} = 0.4 \text{ kg} \quad r = 150 \text{ mm} = 0.15 \text{ m} \]

Shaking force: \[ P = 2 m r \omega_f^2 \sin \omega_f t \]
\[ = (2)(0.4)(0.15)(125.664)^2 \sin \omega_f t \]
\[ = 1.89497 \times 10^3 \sin \omega_f t \]
\[ P_m = 1.89497 \times 10^3 \text{ N} \]

Total mass: \[ M = 140 \text{ kg} \]

By Eqs. (19.48) and (19.52), the vibratory response of the system is
\[ x = x_m \sin(\omega_f t - \varphi) \]

where
\[ x_m = \frac{P_m}{\sqrt{(k - M \omega_f^2)^2 + (c \omega_f)^2}} \quad (1) \]

and
\[ \tan \varphi = \frac{c \omega_f}{k - M \omega_f^2} \quad (2) \]

Since \[ \varphi = 90^\circ = \frac{\pi}{2}, \]
\[ \tan \varphi = \infty \quad \text{and} \quad k - M \omega_f^2 = 0. \]

(a) Combined spring constant,
\[ k = M \omega_f^2 \]
\[ = (140)(125.664)^2 \]
\[ = 2.2108 \times 10^6 \text{ N/m} \]
\[ k = 2.21 \text{ MN/m} \]

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PROBLEM 19.146 (Continued)

The observed amplitude is \( x_m = 15 \text{ mm} = 0.015 \text{ m} \)

From Eq. (1):
\[
c = \frac{1}{\omega_f} \left( \frac{P_m}{x_m} \right)^2 (k - M\omega_f)^2 = \frac{P_m}{\omega_f x_m}
\]
\[
= \frac{1.89497 \times 10^3}{(125.664)(0.015)}
\]
\[
= 1.00531 \times 10^3 \text{ N} \cdot \text{s/m}
\]

Critical damping coefficient:
\[
c_c = 2\sqrt{kM}
\]
\[
= 2\sqrt{(2.2108 \times 10^6)(140)}
\]
\[
= 35.186 \times 10^3 \text{ N} \cdot \text{s/m}
\]

(b) Damping factor.
\[
\frac{c}{c_c} = 35.186 \times 10^3
\]
\[
\frac{c}{c_c} = 0.0286 \quad \blacktriangle
\]
PROBLEM 19.147

A simplified model of a washing machine is shown. A bundle of wet clothes forms a mass $m_b$ of 10 kg in the machine and causes a rotating unbalance. The rotating mass is 20 kg (including $m_b$) and the radius of the washer basket $e$ is 25 cm. Knowing the washer has an equivalent spring constant $k = 1000$ N/m and damping ratio $\zeta = c/c_c = 0.05$ and during the spin cycle the drum rotates at 250 rpm, determine the amplitude of the motion and the magnitude of the force transmitted to the sides of the washing machine.

SOLUTION

Forced circular frequency:
$$\omega_f = \frac{(2\pi)(250)}{60} = 26.18 \text{ rad/s}$$

System mass:
$$m = 20 \text{ kg}$$

Spring constant:
$$k = 1000 \text{ N/m}$$

Natural circular frequency:
$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{20}} = 7.0711 \text{ rad/s}$$

Critical damping constant:
$$c_c = 2\sqrt{km} = 2\sqrt{(1000)(20)} = 282.84 \text{ N \cdot s/m}$$

Damping constant:
$$c = \left(\frac{c}{c_c}\right)c = (0.05)(141.42) = 14.1421 \text{ N \cdot s/m}$$

Unbalance force:
$$P_m = m_b\omega_f^2$$
$$P_m = (10 \text{ kg})(0.25 \text{ m})(26.18 \text{ rad/s})^2 = 1713.48 \text{ N}$$

The differential equation of motion is
$$m\ddot{x} + c\dot{x} + kx = P_m \sin \omega_f t$$

The steady state response is
$$x = x_m \sin(\omega_f t - \varphi) \quad \dot{x} = \omega_f x_m \cos(\omega_f t - \varphi)$$

where
$$x_m = \frac{P_m}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}}$$
$$= \frac{1713.48}{\sqrt{[1000 - (20)(26.18)^2]^2 + [(141421)(26.18)]^2}}$$
$$= \frac{1713.48}{\sqrt{(-12,707.8)^2 + (370.24)^2}} = \frac{1713.48}{12,713.2} = 0.13,478 \text{ m}$$

$$x_m = 134.8 \text{ mm}$$
PROBLEM 19.147 (Continued)

(a) Amplitude of vibration. 
\[ x = 0.13478 \sin(\omega_j t - \varphi) \] 
\[ \dot{x} = (26.18)(0.13478) \cos(\omega_j t - \varphi) \] 
\[ = 3.5285 \cos(\omega_j t - \varphi) \]

Spring force: 
\[ kx = (1000)(0.13478) \sin(\omega_j t - \varphi) \] 
\[ = 134.78 \sin(\omega_j t - \varphi) \]

Damping force: 
\[ c\ddot{x} = (14.1241)(3.5285) \cos(\omega_j t - \varphi) \] 
\[ = 49.901 \cos(\omega_j t - \varphi) \]

(b) Total force: 
\[ F = 134.78 \sin(\omega_j t - \varphi) + 49.901 \cos(\omega_j t - \varphi) \]

Let 
\[ F = F_m \cos \psi \sin(\omega_j t - \varphi) + F_m \sin \varphi \sin(\omega_j t - \varphi) \] 
\[ = F_m \sin(\omega_j t - \varphi + \psi) \]

Maximum force. 
\[ F_m^2 = F_m^2 \cos^2 \psi + F_m^2 \sin^2 \psi \] 
\[ = (134.78)^2 + (49.901)^2 \] 
\[ = 20,656 \]
\[ F_m = 143.7 \text{ N} \]
**PROBLEM 19.148**

A machine element is supported by springs and is connected to a dashpot as shown. Show that if a periodic force of magnitude \( P = P_m \sin \omega_f t \) is applied to the element, the amplitude of the fluctuating force transmitted to the foundation is

\[
F_m = P_m \left[ 1 + \frac{2 \left( \frac{\omega_f}{\omega_n} \right)^2}{\left[ 1 - \left( \frac{\omega_f}{\omega_n} \right)^2 \right]^2 + \left[ \frac{2 \left( \frac{\omega_f}{\omega_n} \right)^2}{1 - \left( \frac{\omega_f}{\omega_n} \right)^2} \right]^2} \right]^{1/2}
\]

**SOLUTION**

From Equation (19.48), the motion of the machine is

\[ x = x_m \sin(\omega_f t - \phi) \]

The force transmitted to the foundation is

Springs:

\[ F_s = kx = kx_m \sin(\omega_f t - \phi) \]

Dashpot:

\[ F_D = c\dot{x} = c x_m \omega_f \cos(\omega_f t - \phi) \]

\[ F_T = x_m \left[ k \sin(\omega_f t - \phi) + c \omega_f \cos(\omega_f t - \phi) \right] \]

or recalling the identity,

\[ A \sin \gamma + B \cos \gamma = \sqrt{A^2 + B^2} \sin(\gamma + \psi) \]

\[ \sin \psi = \frac{B}{\sqrt{A^2 + B^2}} \]

\[ \cos \psi = \frac{A}{\sqrt{A^2 + B^2}} \]

\[ F_T = x_m \sqrt{k^2 + (c \omega_f)^2} \sin(\omega_f t - \phi + \psi) \]

Thus, the amplitude of \( F_T \) is

\[ F_m = x_m \sqrt{k^2 + (c \omega_f)^2} \quad (1) \]

From Equation (19.53):

\[ x_m = \frac{P_m}{k} \sqrt{\left[ 1 - \left( \frac{\omega_f}{\omega_n} \right)^2 \right] + \left[ \frac{2 \left( \frac{\omega_f}{\omega_n} \right)^2}{1 - \left( \frac{\omega_f}{\omega_n} \right)^2} \right]^2} \]

Substituting for \( x_m \) in Equation (1),

\[ F_m = \frac{P_m \left[ 1 + \left( \frac{c \omega_f}{k} \right)^2 \right]}{k \sqrt{\left[ 1 - \left( \frac{\omega_f}{\omega_n} \right)^2 \right] + \left[ \frac{2 \left( \frac{\omega_f}{\omega_n} \right)^2}{1 - \left( \frac{\omega_f}{\omega_n} \right)^2} \right]^2}} \]

\[ \omega_n^2 = \frac{k}{m} \]
PROBLEM 19.148 (Continued)

and Equation (19.41),

\[ c_c = 2m\omega_n \]
\[ m = \frac{c\omega_n}{2} \]
\[ \frac{c\omega_f}{k} = \frac{c\omega_f}{m\omega_n^2} = 2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right) \]

Substituting in Eq. (2),

\[ F_m = \frac{P_m}{\sqrt{1 + \left[ 2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right) \right]^2}} \]

Q.E.D.
PROBLEM 19.149

A 100-kg machine element supported by four springs, each of constant $k = 200 \text{ N/m}$, is subjected to a periodic force of frequency $0.8 \text{ Hz}$ and amplitude $100 \text{ N}$. Determine the amplitude of the fluctuating force transmitted to the foundation if (a) a dashpot with a coefficient of damping $c = 420 \text{ N} \cdot \text{s/m}$ is connected to the machine element and to the ground, (b) the dashpot is removed.

SOLUTION

Forcing frequency: $\omega_f = 2\pi f_f = (2\pi)(0.8) = 1.6\pi \text{ rad/s}$

Exciting force amplitude: $P_m = 100 \text{ N}$

Mass: $m = 100 \text{ kg}$

Equivalent spring constant: $k = (4)(200 \text{ N/m}) = 800 \text{ N/m}$

Natural frequency: $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{800}{100}} = 2.8284 \text{ rad/s}$

Frequency ratio: $\frac{\omega_f}{\omega_n} = \frac{1.6\pi}{2.8284} = 1.77715$

Critical damping coefficient: $c_c = 2\sqrt{km} = 2\sqrt{(800)(100)} = 565.685 \text{ N} \cdot \text{s/m}$

From the derivation given in Problem 19.148, the amplitude of the force transmitted to the foundation is

$$F^*_m = P_m \sqrt{1 + \left[2 \left(\frac{c_c}{\omega_n}\right) \left(\frac{\omega_f}{\omega_n}\right)\right]^2}$$

$$\sqrt{1 - \left(\frac{\omega_f}{\omega_n}\right)^2} + \left[2 \left(\frac{c_c}{\omega_n}\right) \left(\frac{\omega_f}{\omega_n}\right)\right]^2}$$

$$1 - \left(\frac{\omega_f}{\omega_n}\right)^2 = 1 - (1.77715)^2 = -2.1583$$

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PROBLEM 19.149 (Continued)

(a) \( F_m \) when \( c = 420 \text{ N} \cdot \text{s/m} \):

\[
\frac{c}{c_c} = \frac{420}{565.685} = 0.74263
\]

\[
2\left(\frac{c}{c_c}\right)\left(\frac{\omega_f}{\omega_n}\right) = (2)(0.74263)(1.77715)
\]

\[
= 2.63894
\]

From Eq. (1):

\[
F_m = \frac{100\sqrt{1 + (2.63894)^2}}{\sqrt{(-2.1583)^2 + (2.63894)^2}}
\]

\[
= \frac{100\sqrt{7.964}}{\sqrt{11.6223}}
\]

\[
F_m = 82.8 \text{ N} \quad \Box
\]

(b) \( F_m \) when \( c = 0 \):

\[
F_m = \frac{P_m}{1 - (\frac{\omega_f}{\omega_n})^2} = \frac{100}{2.1583}
\]

\[
F_m = 46.3 \text{ N} \quad \Box
PROBLEM 19.150*

For a steady-state vibration with damping under a harmonic force, show that the mechanical energy dissipated per cycle by the dashpot is \( E = \pi c x_m^2 \omega_f \), where \( c \) is the coefficient of damping, \( x_m \) is the amplitude of the motion, and \( \omega_f \) is the circular frequency of the harmonic force.

SOLUTION

Energy is dissipated by the dashpot.

From Equation (19.48), the deflection of the system is

\[ x = x_m \sin(\omega_f t - \phi) \]

The force on the dashpot.

\[ F_D = c\dot{x} \]

\[ F_D = cx_m \omega_f \cos(\omega_f t - \phi) \]

The work done in a complete cycle with

\[ c_f = \frac{2\pi}{\omega_f} \]

\[ E = \int_0^{2\pi/\omega_f} F_D dx \text{ (i.e., force } \times \text{ distance)} \]

\[ dx = x_m \omega_f \cos(\omega_f t - \phi) dt \]

\[ E = \int_0^{2\pi/\omega_f} c x_m^2 \omega_f^2 \cos^2(\omega_f t - \phi) dt \]

\[ \cos^2(\omega_f t - \phi) = \frac{1 - 2\cos(\omega_f t - \phi)}{2} \]

\[ E = c x_m^2 \omega_f^2 \int_0^{2\pi/\omega_f} \frac{1 - 2\cos(\omega_f t - \phi)}{2} dt \]

\[ E = c x_m^2 \omega_f^2 \left[ t - \frac{2\sin(\omega_f t - \phi)}{\omega_f} \right]_0^{2\pi/\omega_f} \]

\[ E = c x_m^2 \omega_f^2 \left[ \frac{2\pi}{\omega_f} - \frac{2}{\omega_f} (\sin(2\pi - \phi) - \sin(\phi)) \right] \]

\[ E = \pi c x_m^2 \omega_f \quad \text{Q.E.D.} \]

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PROBLEM 19.151*

The suspension of an automobile can be approximated by the simplified spring-and-dashpot system shown. (a) Write the differential equation defining the vertical displacement of the mass \( m \) when the system moves at a speed \( v \) over a road with a sinusoidal cross section of amplitude \( \delta_m \) and wavelength \( L \). (b) Derive an expression for the amplitude of the vertical displacement of the mass \( m \).

SOLUTION

(a)

\[
\sum F = ma: \quad W - k(x - \delta) - c \left( \frac{dx}{dt} - \frac{d\delta}{dt} \right) = m \frac{d^2x}{dt^2}
\]

Recalling that \( W = k\delta_{ST} \), we write

\[
m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = k\delta + c \frac{d\delta}{dt}
\]

(1)

Motion of wheel is a sine curve, \( \delta = \delta_m \sin \omega_f t \). The interval of time needed to travel a distance \( L \) at a speed \( v \) is \( t = \frac{L}{v} \), which is the period of the road surface.

Thus,

\[
\omega_f = \frac{2\pi}{T_f} = \frac{2\pi}{\frac{L}{v}} = \frac{2\pi v}{L}
\]

and

\[
\delta = \delta_m \sin \omega_f t \quad \frac{d\delta}{dt} = \frac{\delta_m 2\pi}{L} \cos \omega_f t
\]

Thus, Equation (1) is

\[
m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = (k \sin \omega_f t + c \omega_f \cos \omega_f t)\delta_m
\]
PROBLEM 19.151* (Continued)

(b) From the identity
\[ A \sin y + B \cos y = \sqrt{A^2 + B^2} \sin(y + \psi) \]

\[ \sin \psi = \frac{B}{\sqrt{A^2 + B^2}} \]

\[ \cos \psi = \frac{A}{\sqrt{A^2 + B^2}} \]

We can write the differential equation
\[ m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = \delta m \sqrt{k^2 + (c \omega_f)^2} \sin(\omega_f t + \psi) \]

\[ \psi = \tan^{-1} \frac{c \omega_f}{k} \]

The solution to this equation is analogous to Equations 19.47 and 19.48, with
\[ P_m = \delta m \sqrt{k^2 + (c \omega_f)^2} \]

\[ x = x_m \sin(\omega_f t - \varphi + \psi) \] (where analogous to Equations (19.52))

\[ x_m = \frac{\delta \sqrt{k^2 + (c \omega_f)^2}}{\sqrt{(k - m \omega_f^2)^2 + (c \omega_f)^2}} \]

\[ \tan \varphi = \frac{c \omega_f}{k - m \omega_f^2} \]

\[ \tan \psi = \frac{c \omega_f}{k} \]
PROBLEM 19.152*

Two blocks $A$ and $B$, each of mass $m$, are supported as shown by three springs of the same constant $k$. Blocks $A$ and $B$ are connected by a dashpot, and block $B$ is connected to the ground by two dashpots, each dashpot having the same coefficient of damping $c$. Block $A$ is subjected to a force of magnitude $P = P_m \sin \omega_f t$. Write the differential equations defining the displacements $x_A$ and $x_B$ of the two blocks from their equilibrium positions.

SOLUTION

Since the origins of coordinates are chosen from the equilibrium position, we may omit the initial spring compressions and the effect of gravity

For load $A$,

\[ \Sigma F = ma_A: \quad P_m \sin \omega_f t + 2k(x_B - x_A) + c(\dot{x}_B - \dot{x}_A) = m\ddot{x}_A \]  \hspace{1cm} (1)

For load $B$,

\[ \Sigma F = ma_B: \quad -2k(x_B - x_A) - c(\dot{x}_B - \dot{x}_A) - kx_B - 2c\dot{x}_B = m\ddot{x}_B \]  \hspace{1cm} (2)

Rearranging Equations (1) and (2), we find:

\[ m\ddot{x}_A + c(\dot{x}_A - \dot{x}_B) + 2k(x_A - x_B) = P_m \sin \omega_f t \]
\[ m\ddot{x}_B + 3c\dot{x}_B - c\dot{x}_A + 3kx_B - 2kx_A = 0 \]
PROBLEM 19.153

Express in terms of \( L, C, \) and \( E \) the range of values of the resistance \( R \) for which oscillations will take place in the circuit shown when switch \( S \) is closed.

SOLUTION

For a mechanical system, oscillations take place if \( c < c_c \) (lightly damped).

But from Equation (19.41),
\[
c_c = 2m \sqrt{\frac{k}{m}} = 2\sqrt{km}
\]

Therefore,
\[
c < 2\sqrt{km}
\]  

From Table 19.2:
\[
c \rightarrow R \quad m \rightarrow L
\]
\[
k \rightarrow \frac{1}{C}
\]

Substituting in Eq. (1) the analogous electrical values in Eq. (2), we find that oscillations will take place if
\[
R < 2\sqrt{\left(\frac{1}{C}\right)(L)}
\]

\[
R < 2\sqrt{\frac{L}{C}}
\]
PROBLEM 19.154

Consider the circuit of Problem 19.153 when the capacitor $C$ is removed. If switch $S$ is closed at time $t = 0$, determine (a) the final value of the current in the circuit, (b) the time $t$ at which the current will have reached $(1 - 1/e)$ times its final value. (The desired value of $t$ is known as the time constant of the circuit.)

SOLUTION

Electrical system

Mechanical system

The mechanical analogue of closing a switch $S$ is the sudden application of a constant force of magnitude $P$ to the mass.

(a) Final value of the current corresponds to the final velocity of the mass, and since the capacitance is zero, the spring constant is also zero

\[ \sum F = ma: \quad P - C \frac{dx}{dt} = m \frac{d^2x}{dt^2} \]  

(1)

Final velocity occurs when

\[ \frac{d^2x}{dt^2} = 0 \]

\[ P - C \left. \frac{dx}{dt} \right|_{t_{\text{final}}} = 0 \quad \left. \frac{dx}{dt} \right|_{t_{\text{final}}} = v_{\text{final}} \]

\[ v_{\text{final}} = \frac{P}{C} \]

From Table 19.2: $\quad v \to L, \quad P \to E, \quad C \to R$

Thus,

\[ L_{\text{final}} = \frac{E}{R} \]
PROBLEM 19.154 (Continued)

(b) Rearranging Equation (1), we have

\[ m \frac{d^2x}{dt^2} + C \frac{dx}{dt} = P \]

Substitute

\[ \frac{dx}{dt} = Ae^{-\lambda t} + \frac{P}{C}; \quad \frac{d^2x}{dt^2} = -A\lambda e^{-\lambda t} \]

\[ m \left[ -A\lambda e^{-\lambda t} \right] + C \left[ Ae^{-\lambda t} + \frac{P}{C} \right] = P \]

\[-m\lambda + C = 0 \quad \lambda = \frac{c}{m} \]

Thus,

\[ \frac{dx}{dt} = Ae^{-(c/m)t} + \frac{P}{c} \]

At \( t = 0 \),

\[ \frac{dx}{dt} = 0 \quad 0 = A + \frac{P}{C} \quad A = -\frac{P}{C} \]

\[ v = \frac{dx}{dt} = \frac{P}{c} \left[ 1 - e^{-(c/m)t} \right] \]

From Table 19.2:

\[ v \rightarrow c, \ p \rightarrow E, \ c \rightarrow R, \ m \rightarrow L \]

\[ L = \frac{E}{R} \left[ 1 - e^{-(R/L)t} \right] \]

For \( L = \left( \frac{E}{R} \right) \left( 1 - \frac{1}{e} \right) \),

\[ \left( \frac{R}{L} \right) t = 1 \]

\( t = \frac{L}{R} \)
PROBLEM 19.155

Draw the electrical analogue of the mechanical system shown. (Hint: Draw the loops corresponding to the free bodies \( m \) and \( A \).)

\[ P = P_m \sin \omega_f t \]

SOLUTION

We note that both the spring and the dashpot affect the motion of Point \( A \). Thus, one loop in the electrical circuit should consist of a capacitor \( k \Rightarrow \frac{1}{C} \) and a resistance \( (c \Rightarrow R) \).

The other loop consists of \( (P_m \sin \omega_f t \Rightarrow E_m \sin \omega_f t) \), an inductor \( (m \Rightarrow L) \) and the resistor \( (c \Rightarrow R) \).

Since the resistor is common to both loops, the circuit is
PROBLEM 19.156

Draw the electrical analogue of the mechanical system shown. (*Hint:* Draw the loops corresponding to the free bodies \( m \) and \( A \).)

**SOLUTION**

Loop 1 (Point \( A \)): \[ k_1 \rightarrow \frac{1}{c_1}, \ k_2 \rightarrow \frac{1}{c_2}, \ c_1 \rightarrow R_1 \]

Loop 2 (Mass \( m \)): \[ k_2 \rightarrow \frac{1}{c_2}, \ m \rightarrow L, \ c_2 \rightarrow R_2 \]

With \( k_2 \rightarrow \frac{1}{c_2} \) common to both loops,
PROBLEM 19.157

Write the differential equations defining (a) the displacements of the mass \( m \) and of the Point \( A \), (b) the charges on the capacitors of the electrical analogue.

SOLUTION

(a) Mechanical system.

Point \( A \):

\[ \sum F = 0: \quad c \frac{d}{dt} (x_A - x_m) + kx_A = 0 \]

Mass \( m \):

\[ \sum F = ma: \quad c \frac{d}{dt} (x_m - x_A) - P_m \sin \omega t = -m \frac{d^2 x_m}{dt^2} \]

\[ m \frac{d^2 x_m}{dt^2} + c \frac{d}{dt} (x_m - x_A) = P_m \sin \omega t \]

(b) Electrical analogue.

From Table 19.2:

\[ m \rightarrow L \]
\[ c \rightarrow R \]
\[ k \rightarrow \frac{1}{C} \]
\[ x \rightarrow q \]
\[ P \rightarrow E \]
PROBLEM 19.157 (Continued)

Substituting into the results from Part (a), the analogous electrical characteristics,

\[
R \frac{d}{dt}(q_A - q_m) + \left( \frac{1}{C} \right) q_n = 0
\]

\[
L \frac{d^2 q_m}{dt^2} + R \frac{d}{dt}(q_m - q_A) = E_m \sin \omega_f t
\]

Note: These equations can also be obtained by summing the voltage drops around the loops in the circuit of Problem 19.155.
PROBLEM 19.158

Write the differential equations defining (a) the displacements of the mass \( m \) and of the Point \( A \), (b) the charges on the capacitors of the electrical analogue.

SOLUTION

(a) Mechanical system.

\[ \sum F = 0 \]

Point \( A \):

\[ k_1 x_A + c_1 \frac{dx_A}{dt} + k_2 (x_A - x_m) = 0 \]

\[ c_1 \frac{dx_A}{dt} + (k_1 + k_2)x_A - k_2 x_m = 0 \]

Mass \( m \):

\[ \sum F = ma: \]

\[ k_2 (x_A - x_m) - c_2 \frac{dx_m}{dt} = m \frac{d^2 x_m}{dt^2} \]

\[ m \frac{d^2 x_m}{dt^2} + c_2 \frac{dx_m}{dt} + k_2 (x_m - x_A) = 0 \]

(b) Electrical analogue.

Substituting into the results from Part (a) using the analogous electrical characteristics from Table 19.2 (see left),

\[ R_1 \frac{dq_A}{dt} + \left( \frac{1}{C_1} + \frac{1}{C_2} \right) q_A - \frac{1}{C_2} q_m = 0 \]

\[ L \frac{d^2 q_m}{dt^2} + R_2 \frac{dq_m}{dt} + \frac{1}{C_2} (q_m - q_A) = 0 \]
PROBLEM 19.159

A thin square plate of side $a$ can oscillate about an axis $AB$ located at a distance $b$ from its mass center $G$. (a) Determine the period of small oscillations if $b = \frac{1}{2} a$. (b) Determine a second value of $b$ for which the period of small oscillations is the same as that found in Part a.

SOLUTION

Let the plate be rotated through angle $\theta$ about the axis as shown.

\[ \sum M_{AB} = \sum (M_{AB})_{eff}: \quad mgb \sin \theta = -I \alpha - (m\bar{a})(b) \]

Kinematics:

\[ \alpha = \dot{\theta} \]
\[ a_t = b \alpha = b \dot{\theta} \]
\[ \sin \theta = 0 \]

Moment of inertia:

\[ I = \frac{1}{12} ma^2 \]

Then

\[ (I + mb^2)\dot{\theta} + mb\dot{\theta} = 0 \]
\[ \left( \frac{1}{12} a^2 + b^2 \right) \ddot{\theta} + gb \dot{\theta} = 0 \]
\[ \ddot{\theta} + \frac{12gb}{a^2 + 12b^2} \dot{\theta} = 0 \]

Natural circular frequency:

\[ \omega_n = \sqrt{\frac{12gb}{a^2 + 12b^2}} \]

(a) $b = \frac{1}{2} a$:

\[ \omega_n = \sqrt{\frac{16ga}{a^2 + 3a^2}} = \sqrt{\frac{3g}{2a}} \]

Period of vibration:

\[ \tau_n = \frac{2\pi}{\omega_n} \]

\[ \tau_n = 2\pi \sqrt{\frac{2a}{3g}} \]
PROBLEM 19.159 (Continued)

(b) Another value of $b$ giving the same period:

$$\omega_n = \sqrt{\frac{12gb}{a^2 + 12b^2}} = \sqrt{\frac{3g}{2a}}$$

$$\frac{12b}{a^2 + 12b^2} = \frac{3}{2a}$$

$$24ab = 3a^2 + 36b^2$$

$$36b^2 - 24ab + 3a^2 = 0$$

$$b = 0.5a \quad \text{and} \quad b = 0.16667a$$

$b = 0.1667a$
**PROBLEM 19.160**

A 150-kg electromagnet is at rest and is holding 100 kg of scrap steel when the current is turned off and the steel is dropped. Knowing that the cable and the supporting crane have a total stiffness equivalent to a spring of constant 200 kN/m, determine (a) the frequency, the amplitude, and the maximum velocity of the resulting motion, (b) the minimum tension which will occur in the cable during the motion, (c) the velocity of the magnet 0.03 s after the current is turned off.

**SOLUTION**

![Diagram of electromagnet and cable system]

**Data:**

\[ m_1 = 150 \text{ kg} \quad m_2 = 100 \text{ kg} \quad k = 200 \times 10^3 \text{ N/m} \]

From the first two sketches,

\[ T_0 + kx_m = (m_1 + m_2)g \quad (1) \]

Subtracting Eq. (2) from Eq. (1),

\[ kx_m = m_2g \]

\[ x_m = \frac{m_2g}{k} = \frac{(100)(9.81)}{200 \times 10^3} = 4.905 \times 10^{-3} \text{ m} = 4.91 \text{ mm} \]

Natural circular frequency:

\[ \omega_n = \sqrt{\frac{k}{m_1}} = \sqrt{\frac{200 \times 10^3}{150}} = 36.515 \text{ rad/s} \]

Natural frequency:

\[ f_n = \frac{\omega_n}{2\pi} = \frac{36.515}{2\pi} = 5.81 \text{ Hz} \]

Maximum velocity:

\[ v_m = \omega_n x_m = (36.515)(4.905 \times 10^{-3}) = 0.1791 \text{ m/s} \]

(a) Resulting motion:

- amplitude \( x_m = 4.91 \text{ mm} \)
- frequency \( f_n = 5.81 \text{ Hz} \)
- maximum velocity \( v_m = 0.1791 \text{ m/s} \)
PROBLEM 19.160 (Continued)

(b) Minimum value of tension occurs when \( x = -x_m \).

\[
T_{\text{min}} = T_0 - kx_m
\]
\[
= m_1g - m_2g
\]
\[
= (m_1 - m_2)g
\]
\[
= (50)(9.81)
\]

The motion is given by

\[
x = x_m \sin(\omega_n t + \varphi)
\]
\[
\dot{x} = \omega_n x_m \cos(\omega_n t + \varphi)
\]

Initially,

\[
x_0 = -x_m \quad \text{or} \quad \sin \varphi = -1
\]
\[
\dot{x}_0 = 0 \quad \text{or} \quad \cos \varphi = 0
\]
\[
\varphi = -\frac{\pi}{2}
\]
\[
\dot{x} = \omega_n x_m \cos \left( \omega_n t - \frac{\pi}{2} \right)
\]

(c) Velocity at \( t = 0.03 \) s.

\[
\omega_n t = (36.515)(0.03) = 1.09545 \text{ rad}
\]
\[
\omega_n t - \varphi = -0.47535 \text{ rad}
\]
\[
\cos(\omega_n t - \varphi) = 0.88913
\]
\[
\dot{x} = (36.515)(4.905 \times 10^{-3})(0.88913)
\]
\[
\dot{x} = 0.1592 \text{ m/s}
\]
PROBLEM 19.161

Disks $A$ and $B$ weigh 15 kg and 6 kg, respectively, and a small 2.5-kg block $C$ is attached to the rim of disk $B$. Assuming that no slipping occurs between the disks, determine the period of small oscillations of the system.

SOLUTION

Small oscillations:

$$ h = r_B (1 - \cos \theta_m) = \frac{r_B \dot{\theta}_m^2}{2} $$

Position ①

$$ r_B \dot{\theta}_B = r_A \dot{\theta}_A $$

$$ T_1 = \frac{1}{2} m_C (r_B \dot{\theta}_m)^2 + \frac{1}{2} I_B \dot{\theta}_m^2 + \frac{1}{2} I_A \left( \frac{r_B}{r_A} \dot{\theta}_m \right)^2 $$

$$ I_B = \frac{m_C r_B^2}{2} $$

$$ I_A = \frac{m_A r_A^2}{2} $$

$$ T_1 = \frac{1}{2} \left[ m_C r_B^2 + \frac{r_B r_B^2}{2} + \left( \frac{r_B}{r_A} \right)^2 \frac{r_B r_B^2}{2} \right] \dot{\theta}_m^2 $$

$$ T_1 = \frac{1}{2} \left[ m_C + \frac{m_B}{2} + \frac{m_A}{2} \right] r_B^2 \dot{\theta}_m^2 $$

$$ V_1 = 0 $$

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PROBLEM 19.161 (Continued)

**Position**

\[ T_2 = 0e \]
\[ V_2 = m_c g h = \frac{m_c r^2 \theta_m^2}{2} \]

Conservation of energy and simple harmonic motion.

\[ T_1 + V_1 = T_2 + V_2 \]
\[ \dot{\theta}_m = \omega_n \theta_m \]

\[
\frac{1}{2} \left[ \left( m_c + \frac{m_B}{2} + \frac{m_A}{2} \right) \right] r_B^2 \omega_n^2 \theta_m^2 + 0 = 0 + \frac{m_c g r_B^2 \theta_m^2}{2}
\]

\[ \omega_n^2 = \frac{m_c}{m_c + \frac{(m_B + m_A)}{2}} \frac{g}{r_B} \]

\[ (2.5)(9.81) \]
\[ 2.5 + \frac{(15 + 6)}{2} \]
\[ (0.15) \]

\[ \omega_n^2 = 12.5769 \text{ s}^{-2} \]

**Period of small oscillations.**

\[ \tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{12.5769}} \]

\[ \tau_n = 1.772 \text{ s} \]

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PROBLEM 19.162

A period of 6.00 s is observed for the angular oscillations of a 120-g gyroscope rotor suspended from a wire as shown. Knowing that a period of 3.80 s is obtained when a 30 mm-diameter steel sphere is suspended in the same fashion, determine the centroidal radius of gyration of the rotor. (Density of steel = 7800 kg/m$^3$.)

SOLUTION

\[ \sum M = \Sigma (M)_{\text{eff}}: \quad -K\theta = \bar{T}\ddot{\theta} \]
\[ \ddot{\theta} + \frac{K}{\bar{T}} \theta = 0 \]
\[ \omega_n^2 = \frac{K}{\bar{T}} \]
\[ \tau = 2\pi \sqrt{\frac{\bar{T}}{K}} \]
\[ K = \frac{4\pi^2 \bar{T}}{\tau^2} \]
\[ \bar{T} = \frac{K\tau^2}{4\pi^2} \]  \hspace{1cm} (1)

For the sphere,
\[ r = \frac{d}{2} = \frac{30}{2} = 15 \text{ mm} = 0.015 \text{ m} \]

Volume:
\[ V_s = \frac{4}{3} \pi r^3 = \frac{4}{3} \pi (0.015)^3 \]
\[ = 1.41372 \times 10^{-5} \text{ m}^3 \]

Mass:
\[ m_s = SV_s \]
\[ = (7800 \text{ kg/m}^3)(1.41372 \times 10^{-5}) \]
\[ = 0.11027 \text{ kg} \]

Moment of inertia:
\[ \bar{T} = \frac{2}{5} m_s r^2 = \frac{2}{5} (0.11027)(0.015)^2 \]
\[ = 9.9243 \times 10^{-6} \text{ kgm}^2 \]

Period:
\[ \tau_s = 3.80 \text{ s} \]
PROBLEM 19.162 (Continued)

From Eq. (2):

\[ K = \frac{4\pi^2(9.9243 \times 10^{-6})}{(3.80)^2} \]

= \(2.7133 \times 10^{-5}\) N \(\cdot\) m/rad

For the rotor,

\[ m = 120\,\text{g} = 0.12\,\text{kg} \]
\[ \tau = 6.00\,\text{s} \]

From Eq. (3):

\[ \bar{T} = (2.7133 \times 10^{-5}) \frac{(6.00)^2}{4\pi^2} \]

= \(2.4742 \times 10^{-5}\) kg \(\cdot\) m²

Radius of gyration,

\[ \bar{k} = \sqrt{\frac{\bar{T}}{m}} \]

\[ \bar{k} = \sqrt{\frac{2.4742 \times 10^{-5}}{0.12}} \]

= 0.014359 m

\(\bar{k} = 14.36\,\text{mm}\)
PROBLEM 19.163

A 1.5-kg block B is connected by a cord to a 2-kg block A, which is suspended from a spring of constant 3 kN/m. Knowing that the system is at rest when the cord is cut, determine (a) the frequency, the amplitude, and the maximum velocity of the resulting motion, (b) the minimum tension that will occur in the spring during the motion, (c) the velocity of block A 0.3 s after the cord has been cut.

SOLUTION

Before the cord is cut, the tension in the spring is

\[ T_0 = (m_A + m_B)g = (2 + 1.5)(9.81) = 34.335 \text{ N} \]

The elongation of the spring is

\[ e_0 = \frac{T_0}{k} = \frac{34.335}{3 \times 10^3} = 11.445 \times 10^{-3} \text{ m} \]

After the cord is cut, the tension in the equilibrium position is

\[ T_0' = m_A g = (2.0)(9.81) = 19.62 \text{ N} \]

The corresponding elongation is

\[ e_0' = \frac{T_0'}{k} = \frac{19.62}{3 \times 10^3} = 6.54 \times 10^{-3} \text{ m} \]

Let \( x \) be measured downward from the equilibrium position.

\[ T = T_0' + kx \]

(a) **Newton’s Second Law after the cord is cut.**

\[ m\ddot{x} = m_A g - T \]

\[ m_A \ddot{x} + m_A g - kx - T_0 = -kx \]

\[ m_A \ddot{x} + kx = 0 \]

\[ \omega_n = \sqrt{\frac{k}{m_A}} = \sqrt{\frac{3 \times 10^3}{2}} = 38.7298 \text{ rad/s} \]

\[ f_n = \frac{\omega_n}{2\pi} = \frac{38.7298}{2\pi} \]

\[ f_n = 6.16 \text{ Hz} \]
PROBLEM 19.163 (Continued)

The resulting motion is

\[ x = x_m \sin(\omega_n t + \varphi) \]
\[ \dot{x} = \omega_n x_m \cos(\omega_n t + \varphi) \]

Initial condition:

\[ x_0 = e_0 - e_0' \]
\[ = 4.905 \times 10^{-3} \text{ m} \]
\[ \dot{x}_0 = 0 \]
\[ 0.75375 \times 10^{-3} = x_m \sin \varphi \]
\[ 0 = \omega_n x_m \cos \varphi \]
\[ \varphi = \frac{\pi}{2} \]
\[ x_m = 4.905 \text{ m} \]
\[ x = 4.905 \times 10^{-3} \sin \left( \omega_n t + \frac{\pi}{2} \right) \text{ (m)} \]
\[ \dot{x} = 0.18997 \cos \left( \omega_n t + \frac{\pi}{2} \right) \text{ (m/s)} \]
\[ \dot{x}_m = 0.1900 \text{ m/s} \]

(b) Minimum tension occurs when \( x \) is minimum.

\[ T_{\min} = T_0' - kx_m \]
\[ = 19.62 - (3 \times 10^3)(4.905 \times 10^{-3}) \]
\[ T_{\min} = 4.91 \text{ N} \]

(c) Velocity when \( t = 0.3 \text{ s} \).

\[ \omega_n t - \varphi = (38.7298)(0.3) + \frac{\pi}{2} \]
\[ = 13.1897 \text{ radians} \]
\[ \dot{x} = (0.18997)\cos(13.1897) \]
\[ = (0.18997)(0.8119) \]
\[ \dot{x} = 0.1542 \text{ m/s} \]
PROBLEM 19.164

Two rods, each of mass \( m \) and length \( L \), are welded together to form the assembly shown. Determine (a) the distance \( b \) for which the frequency of small oscillations of the assembly is maximum, (b) the corresponding maximum frequency.

SOLUTION

Position ①

\[
V_1 = 0
\]

\[
T_1 = \frac{1}{2} \left[ m \bar{v}_{CD}^2 + m \bar{v}_{AB}^2 + \bar{T}_{CD} \dot{\theta}_m^2 + \bar{T}_{AB} \dot{\theta}_m^2 \right]
\]

\[
\bar{v}_{CD} = b \dot{\theta}_m
\]

\[
\bar{v}_{AB} = \left( \frac{L}{2} \right) \dot{\theta}_m
\]

\[
\bar{T}_{CD} = \bar{T}_{AB}
\]

\[
= \frac{1}{12} mL^2
\]

\[
T_1 = \frac{1}{2} \left[ b^2 + \left( \frac{L}{2} \right)^2 + \frac{1}{12} L^2 + \frac{1}{12} L^2 \right] \dot{\theta}_m^2
\]

\[
= \frac{m}{2} \left( b^2 + \frac{5L^2}{12} \right) \dot{\theta}_m^2
\]
PROBLEM 19.164 (Continued)

Position

\[
V_2 = mgb(1 - \cos \theta_m) + mg \frac{L}{2}(1 - \cos \theta_m)
\]

Small angles:

\[
1 - \cos \theta_m = 2 \sin^2 \left( \frac{\theta_m}{2} \right) = \frac{\omega_m^2}{2}
\]

\[
V_2 = mg \frac{\omega_m^2}{2}(b + \frac{L}{2})
\]

\[
T_2 = 0
\]

Conservation of energy and simple harmonic motion.

\[
T_1 + V_1 = T_2 + V_2
\]

\[
\frac{1}{2}m \left[ b^2 + \frac{5}{12}L^2 \right] \omega_m^2 + 0 = 0 + \frac{mg}{2} \left[ b + \frac{L}{2} \right] \omega_m^2
\]

\[
\omega_m^2 = \frac{g(b + \frac{L}{2})}{\left( b^2 + \frac{5}{12}L^2 \right)}
\]

(a) Distance \( b \) for maximum frequency.

Maximum \( \omega_m^2 \) when \( \frac{d\omega_m^2}{db} = 0 \)

\[
\frac{d\omega_m^2}{db} = \left( b^2 + \frac{5}{12}L^2 \right) g - g \left( b + \frac{L}{2} \right) (2b) = 0
\]

\[
-b^2 - Lb + \left( \frac{5}{12} \right) L^2 = 0
\]

\[
b = -L \pm \sqrt{L^2 + \left( \frac{20}{12} \right) L^2} = 0.316L, 1.317L
\]

\( b = 0.316L \)

(b) Corresponding maximum frequency.

From Eq. (1) and the answer to Part (a):

\[
\omega_n^2 = \frac{g[0.316 + 0.5]}{(0.316)^2 + \frac{5}{12}L}
\]

\[
= 1.580 \frac{g}{L}
\]

\[
f_n = \frac{\sqrt{\omega_n}}{2\pi} = \sqrt{\frac{1.580}{2\pi}} \sqrt{\frac{g}{L}}
\]

\[
= 0.200 \sqrt{\frac{g}{L}} \text{ Hz}
\]
PROBLEM 19.165

As the rotating speed of a spring-supported motor is slowly increased from 200 to 500 rpm, the amplitude of the vibration due to the unbalance of the rotor is observed to decrease steadily from 8 mm to 2.5 mm. Determine (a) the speed at which resonance would occur, (b) the amplitude of the steady-state vibration at a speed of 100 rpm.

SOLUTION

Since the amplitude decreases with increasing speed over the range $\omega_1 = 200$ rpm to $\omega_2 = 500$ rpm, the motion is out of phase with the force.

$$x_m = \frac{P_c}{k} \left( 1 - \left( \frac{\omega_1}{\omega_n} \right)^2 \right) = \frac{m\omega_1^2}{k} \left( 1 - \left( \frac{\omega_1}{\omega_n} \right)^2 \right) = \frac{m\omega_f^2}{k} \left( 1 - \left( \frac{\omega_f}{\omega_n} \right)^2 \right)$$

$$|x_m| = \left( \frac{m}{M} \right) \left( \frac{\omega_f}{\omega_n} \right)^2 \left( \omega_1^2 - 1 \right)$$

Let $u = \frac{\omega_1}{\omega_n}$, where $\omega_1 = 200$ rpm = 20.944 rad/s.

$$|x_m| = \left( \frac{m}{M} \right) u^2 \left( u^2 - 1 \right)$$

At $\omega_f = \omega_2 = 500$ rpm

$$\frac{\omega_f}{\omega_n} = \frac{500}{200} = \frac{5}{2} = 2.5u$$

$$|x_m| = \frac{m}{M} \left( \frac{\omega_f}{\omega_n} \right)^2 \left( \omega_1^2 - 1 \right)$$

$$|x_m| = \left( \frac{m}{M} \right) \left( \frac{5}{2} \right)^2 \left( 20.944^2 - 1 \right)$$

$$|x_m| = \frac{m}{M} \left( \frac{25}{4} \right) \left( 434.976 - 1 \right)$$

$$|x_m| = 62.5 \left( 433.976 \right)$$

$$|x_m| = 27.5 \text{ mm}$$

At 100 rpm

$$\frac{\omega_f}{\omega_n} = \frac{100}{200} = \frac{1}{2} = 0.5u$$

$$|x_m| = \frac{m}{M} \left( \frac{1}{2} \right)^2 \left( 20.944^2 - 1 \right)$$

$$|x_m| = \frac{1}{4} \left( 434.976 - 1 \right)$$

$$|x_m| = 10.8 \text{ mm}$$

At $62.5u^4 - u^2 = 20u^4 - 20u^2$

$$13.75u^4 - 19u^2 = 0$$

$$u = \sqrt[4]{19} = 1.17551$$

(a) Natural frequency

$$\omega_n = \frac{\omega_1}{u} = \frac{20.944}{1.17551} = 17.817 \text{ rad/s}$$

$\omega_n = 170.1 \text{ rpm}$
PROBLEM 19.165 (Continued)

At 200 rpm, \( |x_m| = 8 \text{ mm} = 8 \times 10^{-3} \text{ m}, \quad \omega = 1.17551 \)

From Eq. (1):

\[
8 \times 10^{-3} = \frac{\left(\frac{mr}{M}\right)(1.17551)^2}{(1.17551)^2 - 1}
\]

\[
= \frac{19}{5.25} \frac{mr}{M}
\]

\[
\frac{mr}{M} = 2.2105 \times 10^{-3} \text{ m}
\]

(b) Amplitude at 100 rpm.

\[
\omega_f = 10.472 \text{ rad/s} < \omega_n
\]

\[
\frac{\omega_f}{\omega_n} = \frac{10.472}{17.817} = 0.58775
\]

The motion is in phase with the force.

\[
x_m = \frac{\left(\frac{mr}{M} \frac{\omega_f}{\omega_n}\right)^2}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2}
\]

\[
x_m = \frac{(2.2105 \times 10^{-3})(0.58775)^2}{1 - (0.58775)^2}
\]

\[
x_m = 1.167 \times 10^{-3} \text{ m}
\]

\[
x_m = 1.167 \text{ mm}
\]
PROBLEM 19.166

The compressor shown has a mass of 250 kg and operates at 2000 rpm. At this operating condition, undesirable vibration occurs when the compressor is attached directly to the ground. To reduce the vibration of the concrete floor that is resting on clay soil, it is proposed to isolate the compressor by mounting it on a square concrete block separated from the rest of the floor as shown. The density of concrete is 2400 kg/m³ and the spring constant for the soil is found to be $80 \times 10^6$ N/m. The geometry of the compressor leads to choosing a block that is 1.5 m by 1.5 m. Determine the depth $h$ that will reduce the force transmitted to the ground by 75%.

SOLUTION

Forced circular frequency corresponding to 2000 rpm:

$$\omega_f = \frac{(2\pi)(2000)}{60} = 209.44 \text{ rad/s}$$

Natural circular frequency for compressor attached directly to the ground:

$$\omega_1 = \sqrt{\frac{k}{m}} = \sqrt{\frac{80 \times 10^6 \text{ N/m}}{250 \text{ kg}}} = 565.68 \text{ rad/s}$$

Let $P$ be the force due to the compressor unbalance and $F_1$ be the force transmitted to the ground.

$$F_1 = kx_1 = k \cdot \frac{P}{\omega_f^2} = \frac{P}{1 - \left(\frac{\omega_1}{\omega_f}\right)^2}$$

$$F_1 = \frac{1}{1 - \left(\frac{\omega_1}{\omega_f}\right)^2} = \frac{1}{1 - \left(\frac{209.44}{565.68}\right)^2} = 1.15951$$
PROBLEM 19.166 (Continued)

Modify the system by making the mass much larger so that the natural frequency of modified system is much less than the forcing frequency. Let \( \omega_2 \) be the new natural circular frequency. For the ground force to be reduced by 75%,

\[
\frac{F_2}{F_1} = 0.25 = (0.25)(1.15951) = 0.28988 = \frac{1}{\left(\frac{\omega_f}{\omega_c}\right)^2 - 1}
\]

\[
\left(\frac{\omega_f}{\omega_2}\right)^2 = 1 + \frac{1}{0.28988} = 4.4497 \quad \frac{\omega_f}{\omega_2} = 2.109
\]

\[
\omega_2^2 = \frac{\omega_f^2}{4.4497} = \frac{(209.44)^2}{4.4497} = 9.8579 \times 10^3 \text{ (rad/s)}^2
\]

\[
\omega_2^2 = \frac{k}{m_2}
\]

\[
m_2 = \frac{k}{\omega_2^2} = \frac{80 \times 10^6 \text{ N/m}}{9.8579 \times 10^3 \text{ (rad/s)}^2} = 8115 \text{ kg}
\]

Required properties of attached concrete block:

- mass \( m_2 - m_1 = 8115 - 250 = 7865 \text{ kg} \)
- volume \( \frac{\text{mass}}{\text{density}} = \frac{7865 \text{ kg}}{2400 \text{ kg/m}^3} = 3.277 \text{ m}^3 \)
- area \( 1.5 \text{ m} \times 1.5 \text{ m} = 2.25 \text{ m}^2 \)
- depth \( \frac{\text{volume}}{\text{area}} = \frac{3.277 \text{ m}^3}{2.25 \text{ m}^2} = 1.456 \text{ m} \)

\( h = 1.456 \text{ m} \)
PROBLEM 19.167

If either a simple or a compound pendulum is used to determine experimentally the acceleration of gravity $g$, difficulties are encountered. In the case of the simple pendulum, the string is not truly weightless, while in the case of the compound pendulum, the exact location of the mass center is difficult to establish. In the case of a compound pendulum, the difficulty can be eliminated by using a reversible, or Kater, pendulum. Two knife edges $A$ and $B$ are placed so that they are obviously not at the same distance from the mass center $G$, and the distance $l$ is measured with great precision. The position of a counterweight $D$ is then adjusted so that the period of oscillation $\tau$ is the same when either knife edge is used. Show that the period $\tau$ obtained is equal to that of a true simple pendulum of length $l$ and that $g = 4\pi^2l/\tau^2$.

SOLUTION

From Problem 19.52, the length of an equivalent simple pendulum is:

$$l_A = \ell + \frac{k^2}{\ell}$$

and

$$l_B = \ell + \frac{k^2}{\ell}$$

But

$$\tau_A = \tau_B$$

$$2\pi \sqrt{\frac{l_A}{g}} = 2\pi \sqrt{\frac{l_B}{g}}$$

Thus,

$$l_A = l_B$$

For

$$\ell + \frac{k^2}{\ell} = \ell + \frac{k^2}{\ell}$$

$$\tau^2 \ell + k^2 \ell = \tau^2 \ell + k^2 \ell$$

$$\tau R (\tau - R) = k^2 [R - \ell]$$

$$(\tau - R) \approx 0$$

Thus,

$$\tau R = k^2$$

or

$$\tau = \frac{k^2}{R}$$

$$\ell = \frac{k^2}{\tau}$$
PROBLEM 19.167 (Continued)

Thus, \[ AG = GA' \] and \[ BG = GB' \]

That is, \[ A = A' \] and \[ B = B' \]

Noting that \[ l_A = l_B = l \]

\[ \tau = 2\pi \sqrt{\frac{l}{g}} \]

or \[ g = \frac{4\pi^2 l}{\tau^2} \]
PROBLEM 19.168

A 400-kg motor supported by four springs, each of constant 150 kN/m, is constrained to move vertically. Knowing that the unbalance of the rotor is equivalent to a 23-g mass located at a distance of 100 mm from the axis of rotation, determine for a speed of 800 rpm (a) the amplitude of the fluctuating force transmitted to the foundation, (b) the amplitude of the vertical motion of the motor.

SOLUTION

Total mass: \[ M = 400 \text{ kg} \]

Unbalance: \[ m' = 23 \text{ g} = 0.023 \text{ kg} \]
\[ r = 100 \text{ mm} = 0.100 \text{ m} \]

Forcing frequency: \[ \omega_f = 800 \text{ rpm} \]
\[ = 83.776 \text{ rad/s} \]

Spring constant: \[ k = (4)(150 \times 10^3 \text{ N/m}) \]
\[ = 600 \times 10^3 \text{ N/m} \]

Natural frequency:
\[ \omega_n = \sqrt{\frac{k}{m}} \]
\[ = \sqrt{\frac{600 \times 10^3}{400}} \]
\[ = 38.730 \text{ rad/s} \]

Frequency ratio:
\[ \frac{\omega_f}{\omega_n} = 2.1631 \]

Unbalance force:
\[ P_m = m' r \omega_f^2 \]
\[ = (0.023)(0.100)(83.776)^2 \]
\[ = 16.1424 \text{ N} \]

Static deflection:
\[ \frac{P_m}{k} = \frac{16.1424}{600 \times 10^3} \]
\[ = 26.904 \times 10^{-6} \text{ m} \]
PROBLEM 19.168 (Continued)

Amplitude of vibration. From Eq. (19.33):

\[ x_m = \frac{\frac{P_m}{k}}{1 - \left(\frac{\omega_f}{\omega_n}\right)^2} \]

\[ = \frac{26.904 \times 10^{-6}}{1 - (2.1631)^2} \]

\[ = -7.313 \times 10^{-6} \text{ m} \]

(a) Transmitted force.

\[ F_m = -kx_m = -(600 \times 10^3)(-7.313 \times 10^{-6}) \quad F_m = 4.39 \text{ N} \uparrow \]

(b) Amplitude of motion.

\[ |x_m| = 7.313 \times 10^{-6} \text{ m} \quad |x_m| = 0.00731 \text{ mm} \uparrow \]
PROBLEM 19.169

Solve Problem 19.168, assuming that a dashpot of constant \( c = 6500 \text{ N} \cdot \text{s/m} \) is introduced between the motor and the ground.

PROBLEM 19.168 A 400-kg motor supported by four springs, each of constant 150 kN/m, is constrained to move vertically. Knowing that the unbalance of the rotor is equivalent to a 23-g mass located at a distance of 100 mm from the axis of rotation, determine for a speed of 800 rpm (a) the amplitude of the fluctuating force transmitted to the foundation, (b) the amplitude of the vertical motion of the motor.

SOLUTION

Total mass: \( M = 400 \text{ kg} \)

Unbalance: \( m = 23 \text{ g} = 0.023 \text{ kg} \)
\( r = 100 \text{ mm} = 0.100 \text{ m} \)

Forcing frequency: \( \omega_f = 800 \text{ rpm} \)
\( = 83.776 \text{ rad/s} \)

Spring constant: \( (4)(150 \times 10^3 \text{ N/m}) = 600 \times 10^3 \text{ N/m} \)

Natural frequency:
\[
\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{600 \times 10^3}{400}} = 38.730 \text{ rad/s}
\]

Frequency ratio:
\[
\frac{\omega_f}{\omega_n} = 2.1631
\]

Viscous damping coefficient: \( c = 6500 \text{ N} \cdot \text{s/m} \)

Critical damping coefficient:
\[
c_c = 2\sqrt{kM} = 2\sqrt{(600 \times 10^3)(400)} = 30,984 \text{ N} \cdot \text{s/m}
\]

Damping factor:
\[
\frac{c}{c_c} = 0.20978
\]

Unbalance force:
\[
P_m = mr\omega_f^2 = (0.023)(0.100)(83.776)^2 = 16.1424 \text{ N}
\]

Static deflection:
\[
\delta_{ST} = \frac{P_m}{k} = \frac{16.1424}{600 \times 10^3} = 26.904 \times 10^{-6} \text{ m}
\]
PROBLEM 19.169 (Continued)

Amplitude of vibration. Use Eq. (19.53).

\[ x_m = \frac{\frac{P_m}{k}}{\sqrt{1 - \left(\frac{\omega_f}{\omega_n}\right)^2 + \left[2\left(\frac{c}{c_n}\right)\left(\frac{\omega_f}{\omega_n}\right)\right]^2}} \]

where

\[ 1 - \left(\frac{\omega_f}{\omega_n}\right)^2 = 1 - (2.1631)^2 = -3.679 \]

and

\[ 2\left(\frac{c}{c_n}\right)\left(\frac{\omega_f}{\omega_n}\right) = (2)(0.20978)(2.1631) = 0.90755 \]

\[ x_m = \frac{26.904 \times 10^{-3}}{\sqrt{(-3.679)^2 + (0.90755)^2}} \]

\[ = 7.1000 \times 10^{-6} \text{ m} \]

Resulting motion:

\[ x = x_m \sin(\omega_f t - \varphi) \]

\[ \dot{x} = \omega_f x_m \cos(\omega_f t - \varphi) \]

Spring force:

\[ F_s = kx = kx_m \sin(\omega_f t - \varphi) = 4.26 \sin(\omega_f t - \varphi) \]

Damping force:

\[ F_d = c\dot{x} = c\omega_f x_m \cos(\omega_f t - \varphi) = 3.8663 \cos(\omega_f t - \varphi) \]

Let

\[ F_s = F_m \cos \psi \sin(\omega_f t - \varphi) \quad \text{and} \quad F_d = F_m \sin \psi \cos(\omega_f t - \varphi) \]

Total force:

\[ F = F_m \cos \psi \sin(\omega_f t - \varphi) + F_m \sin \psi \cos(\omega_f t - \varphi) \]

\[ = F_m \sin(\omega_f t - \varphi + \psi) \]

(a) Force amplitude.

\[ F_m = \sqrt{(F_m \cos \psi)^2 + (F_m \sin \psi)^2} \]

\[ F_m = \sqrt{(kx_m)^2 + (c\omega_f x_m)^2} \]

\[ = \sqrt{(4.26)^2 + (3.8663)^2} \]

\[ F_m = 5.75 \text{ N} \uparrow \]

(b) Amplitude of vibration.

\[ x_m = 0.00710 \text{ mm} \uparrow \]
PROBLEM 19.170

A small ball of mass \( m \) attached at the midpoint of a tightly stretched elastic cord of length \( l \) can slide on a horizontal plane. The ball is given a small displacement in a direction perpendicular to the cord and released. Assuming the tension \( T \) in the cord to remain constant, \((a)\) write the differential equation of motion of the ball, \((b)\) determine the period of vibration.

SOLUTION

\((a)\) Differential equation of motion.

\[ \sum F = ma: \quad 2T \sin \theta = -m \ddot{x} \]

For small \( x \),

\[ \sin \theta = \tan \theta = \frac{x}{(\frac{1}{2})} = \frac{2x}{l} \]

\[ m \ddot{x} + \left( \frac{4T}{l} \right) x = 0 \]

Natural circular frequency.

\[ \omega_n^2 = \frac{4T}{ml} \]

\[ \omega_n = 2\sqrt{\frac{T}{ml}} \]

\((b)\) Period of vibration.

\[ \tau_n = \frac{2 \pi}{\omega_n} \]

\[ \tau_n = \pi \sqrt{\frac{ml}{T}} \]